Department of Mathematics
Stockholm University
Division of Mathematics
Examiner: Samuel Lundqvist

Exam in
MM7025 – Computer algebra
7.5 hp
May 20 2020

You are allowed to use the textbook, Macaulay2, personal notes and everything on the course page. Make sure to justify your answers; *just referring to a Macaulay2 computation is not enough.*

Pass part

1. In $\mathbb{Q}[x, y, z]$, order the monomials $x, y^2z, xz^2, yx, 1$ in decreasing order with respect to $x > y > z$ and
   (a) Lex,
   (b) Graded Reverse Lex. 6p
2. Let $I = \langle x^4, y^5, z^6\rangle \subset \mathbb{Q}[x, y, z]$. Determine if $x^3y^6z^7 + x^2y^6z^7 + x^2y^4z^5 \in I$. 6p
3. In $\mathbb{Q}[x]$, let $I = \langle x^4 + x^2 + 1, x^3 + x\rangle$. Determine $\mathbf{V}(I)$.
4. Let $I = \langle x^2 - xz, y^2 - yx, z^2\rangle \subset \mathbb{Q}[x, y, z]$. Compute a Gröbner basis for I with respect to Lex and $x > y > z$. 6p

Higher degree part

1. Let $I = \langle x^3, y^3, z^3, xyz\rangle \subset \mathbb{Q}[x, y, z]$. Determine the monomials in $\mathbb{Q}[x, y, z]$ which are not in I. 6p
2. In $\mathbb{Q}[x, y]$, determine whether $\langle y^2 + 1, x^2 + y\rangle = \langle x^3 + xy, x^2y - 1\rangle$ holds or not. 6p
3. Determine the generating sets of the ideals $I_1, I_2, I_3 \subset \mathbb{C}[x, y]$, where

   $$I_1 = \mathbf{I}(\{(0, 0)\}), I_2 = \mathbf{I}(\{(0, 0), (0, 1)\}), I_3 = \mathbf{I}(\{(0, 0), (0, 1), (1, 0)\}).$$

   (The first two problems are from the exercise from the lecture notes for day 10.) 6p
4. Let $I = \langle x^2 - x, y^2 - y, z^2 - z, u^2 - u, xy + z + ux + 1\rangle \subset \mathbb{Z}_2[x, y, z, u]$. Determine $\mathbf{V}(I)$. 6p
5. Let I be an ideal in $\Bbbk[x_1, \ldots, x_n]$, $\Bbbk$ a field. Show that $G = \{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis of I with respect to the term order $\prec$ if and only if the leading term of any element of I is divisible by one of the $LT(g_i)$ (with respect to $\prec$). (Exercise 2.5.5.) 6p
6. Let $V = \mathbf{V}(x_3 - x_1^2, x_4 - x_1x_2, x_2x_4 - x_1x_5, x_4^2 - x_3x_5)$.
   - Using any convenient monomial order, determine a collection of monomials spanning the space of remainders modulo a Gröbner basis for the ideal generated by the defining equations of V (over $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]$).
   - For which i is there some $m_i \geq 0$ such that $x_i^{m_i} \in \langle LT(I)\rangle$?
   - Is V a finite set? Why or why not?

   (Exercise 5.3.6.) 6p

GOOD LUCK!