FINAL EXAM

Instructions: Justify your answers. You may use results from the homework sets, but make sure to carefully state such results. No calculators and no notes allowed.

Grading: This exam is worth 30 points. If you completed homework assignments, your homework bonus (out of 3 points) will be added to your score. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: $[26.5, 30]$, B: $[23, 26.5)$, C: $[19.5, 23)$, D: $[16, 19.5)$, E: $[12.5, 16)$, F: $[0, 12.5)$.

Problem 1. *Let $f(x) = x^7 - 20 \in \mathbf{Q}[x]$.*

(a) (1 point) Show that f is irreducible over $\mathbf{Q}$.
(b) (2 points) Give an explicit description of a splitting field L for f over $\mathbf{Q}$.
(c) (1 point) Compute $[L : \mathbf{Q}]$. Justify your answer.
(d) (1 point) Show that $L/\mathbf{Q}$ is Galois.

Problem 2. *Let f and L be as in Problem 1.*

(a) (2 points) Give generators and relations for $\mathrm{Gal}(L/\mathbf{Q})$.
(b) (2 points) Show that $\mathrm{Gal}(L/\mathbf{Q})$ is solvable.
(c) (2 points) Show that there is a unique extension $K/\mathbf{Q}$ of degree 6 which is contained in L.
(d) (2 poins) Show that there is a unique quadratic extension $F/\mathbf{Q}$ contained in L and describe F as $\mathbf{Q}(\sqrt{D})$ for some integer D.

Problem 3. *Let $\Phi_{15}(x) \in \mathbf{Z}[x]$ be the cyclotomic polynomial of primitive 15th roots of unity. Let ζ be a root of $\Phi_{15}(x)$ in some finite extension of $\mathbf{Q}$.*

(a) (2 points) Show that for every prime p, the reduction of $\Phi_{15}(x)$ modulo p is reducible in $\mathbf{F}_p[x]$.
(b) (1 point) Is the regular 15-gon constructible by straightedge and compass? Justify your answer.
(c) (2 point) Show that there are precisely three quadratic extensions of $\mathbf{Q}$ contained in $\mathbf{Q}(\zeta)$.
(d) (2 points) Describe the three distinct quadratic extensions of $\mathbf{Q}$ contained in $\mathbf{Q}(\zeta)$ in the form $\mathbf{Q}(\sqrt{D})$, where $D \in \mathbf{Z}$ is an integer.

Problem 4.

(a) (2 points) Let p be a prime, let $a \in \mathbf{F}_p^\times$ and put $g(x) = x^p - x + a$. Show that $g(x)$ is irreducible in $\mathbf{F}_p[x]$.
(b) (2 points) Let G be a subgroup of S_5 which contains a 5-cycle and a transposition. Show that $G = S_5$.
(c) (2 points) Assume k is an integer which is divisible by 3 and not divisible by 5. Show that the Galois group of $h(x) = x^5 - x + k \in \mathbf{Q}[x]$ is S_5.

Problem 5.

(a) (1 point) Show that $x^4 + x + 1$ divides $x^{16} - x$ in $\mathbf{F}_2[x]$.
(b) (1 point) Show that $x^4 + x + 1$ divides $x^{27} - x$ in $\mathbf{F}_3[x]$.
(c) (1 point) Show that the Galois group of $x^4 + 7x + 1 \in \mathbf{Q}[x]$ is S_4.
(d) (1 point) Let α be a real root of $x^4 + 7x + 1$. Show that α is not constructible by straightedge and compass.

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