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Suggested solutions:
Econometric methods (MT5014)
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Problem 1

Set

$$\boldsymbol{\Omega} = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix}$$

and

$$\boldsymbol{X} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ 1 & X_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Use the formula on p. 90 in Tyrcha et al. (in this case $\sigma^2 = 1$) to find, with tedious but standard calculations,

$$V([\hat{\alpha}_{GLS}, \hat{\beta}_{GLS}]^T|\mathbf{X}) = (\boldsymbol{X}^T\boldsymbol{\Omega}^{-1}\boldsymbol{X})^{-1} = [...] = \begin{bmatrix} 2.5 & -1 \\ -1 & 0.5 \end{bmatrix}.$$

Hence, $Cov(\hat{\alpha}_{GLS}, \hat{\beta}_{GLS}|\boldsymbol{X}) = -1$ and $Var(\hat{\beta}_{GLS}|\boldsymbol{X}) = 0.5$.

Set also

$$\boldsymbol{Y} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}.$$

Using the formula for the GLS estimator (p. 89 in Tyrcha et al.) and using standard calculations we find

$$\begin{bmatrix} \hat{\alpha}_{GLS} \\ \hat{\beta}_{GLS} \end{bmatrix} = (\boldsymbol{X}^T\boldsymbol{\Omega}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{\Omega}^{-1}\boldsymbol{Y} = [...] = \begin{bmatrix} 10.5 \\ -2.5. \end{bmatrix}$$

Problem 2

The OLS estimator can in this case be derived (similarly to how it is done in Tyrcha et al ch. 2) as

$$\hat{\beta} = \frac{\sum X_i Y_i}{\sum X_i^2},$$

so that with $X = (X_1, ..., X_n)^T$ and $Y = (Y_1, ..., Y_n)^T$ it holds that

$$\hat{\beta} = (X^T X)^{-1} X^T Y.$$

Hence, if we also set $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, then $Y = X\beta + \varepsilon$ and

$$\begin{aligned} E(\hat{\beta}) &= E((X^T X)^{-1} X^T Y) \\ &= E((X^T X)^{-1} X^T (X\beta + \varepsilon)) \\ &= \beta, \end{aligned}$$

and $\hat{\beta}$ is unbiased.

To show that $\hat{\beta}$ is BLUE means showing that any other linear unbiased estimator has a larger variance. Let $\hat{\gamma}$ be a linear estimator, meaning that for some column vector C (dimension n), it holds that $\hat{\gamma} = C^T Y$. Suppose moreover that $\hat{\gamma}$ is unbiased so that $E(\hat{\gamma}) = E(C^T Y) = E(C^T (X\beta + \varepsilon)) = C^T X\beta = \beta$, implying that

$$C^T X = \sum C_i X_i = 1. \tag{1}$$

Also,

$$\begin{aligned} V(\hat{\gamma}) &= V(C^T (X\beta + \varepsilon)) \\ &= V(C^T X\beta + C^T \varepsilon) \\ &= V\left(\sum C_i \varepsilon_i\right) \\ &= \sigma^2 \sum C_i^2. \end{aligned} \tag{2}$$

Hence, to find the estimator that is BLUE we simply must the solve problem of minimizing (2) with respect to the variables C_i given the constraint (1). This is a standard constrained optimization problem that is easily solved using the Lagrange multiplier method, which yields $C_i = X_i / \sum X_i^2$; which is directly seen to be equivalent to $\hat{\beta}$ as defined above. In other words, $\hat{\beta}$ is indeed BLUE.

Problem 3

We have the classical model under heteroskedasticity and the GLS estimator is BLUE (Tyrcha et al p. 90). The data corresponds to $\boldsymbol{\Omega} = diag(a_1^2, ..., a_{100}^2)$,

$$\boldsymbol{X} = \begin{bmatrix} 1 & X_{1,2} & X_{2,1} \\ \vdots & \vdots & \vdots \\ 1 & X_{1,100} & X_{2,100} \end{bmatrix}, \boldsymbol{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_{100} \end{bmatrix}$$

Using the same formula as in Problem 1 we can, using the data as described above, calculate

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{GLS}} = (\boldsymbol{X}^T \boldsymbol{\Omega}^{-1} \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{\Omega}^{-1} \boldsymbol{Y}.$$

Using that $n = 100$ and $k = 3$, we have now have all ingredients for the formula for $\hat{\sigma}^2$ in Tyrcha et al. p. 91.

We will use an F-test to test the hypothesis against H_1 : any $\beta_i \neq 0, i = 1, 2$ (cf. Tyrcha et al. ch 3.3), which corresponds to $q = 2$,

$$\boldsymbol{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{r} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We thus have all ingredients for the formula for F (Tyrcha et al. p. 91) which is distributed according to $F(q, n-k)$ under H_0. Using our data, as described above, and the formulas above we may (using the formula for F) calculate F_{obs}. A table gives $F_{0.01}(2, 97) = 4.8309$. If $F_{obs} > 4.8309$ we reject H_0 in favor of H_1.

Problem 4

Repeated substitution gives

$$\begin{aligned} r_t &= 0.3r_{t-1} + a_t \\ &= 0.3^2 r_{t-2} + 0.3a_{t-1} + a_t \\ &= 0.3^3 r_{t-3} + 0.3^2 a_{t-2} + 0.3a_{t-1} + a_t \\ &= ... \\ &= a_t + 0.3a_{t-1} + 0.3^2 a_{t-2} + ... \end{aligned}$$

Hence, $E(r_t) = 0$,

$$\begin{aligned} V(r_t) &= V(a_t + 0.3a_{t-1} + 0.3^2 a_{t-2} + ...) \\ &= \sum_{i=0}^{\infty} (0.3^2)^i \\ &= \frac{1}{1-0.3^2}, \end{aligned}$$

and (using e.g. $E(r_t) = 0$ and the independence of a_{t+1} and r_t)

$$\begin{aligned} C(r_t, r_{t+1}) &= E(r_t r_{t+1}) \\ &= E(r_t(0.3r_t + a_{t+1})) \\ &= 0.3E(r_t^2) \\ &= 0.3V(r_t) \\ &= \frac{0.3}{1-0.3^2} \end{aligned}$$

while similar calculations yield $C(r_t, r_{t+L}) = \frac{0.3^L}{1-0.3^2}$ for $L > 1$. Since the expectation, variance and covariances are independent of t, the time series is weakly stationary.

Problem 5

Define

$$I_{t-1} = \begin{cases} 0.4 & \text{if } r_{t-1} \geq 1, \\ 0.2 & \text{if } r_{t-1} < 1, \end{cases}$$

so that the model can be written as $r_t = I_{t-1}r_{t-1} + a_t$.

Note that $I_1 = 0.4$ so that $r_2 = I_1 r_1 + a_2 = 0.4 + a_2$. Hence, the 1 step ahead forecast is

$$\hat{r}_1(1) = E[r_2 | F_1] = 0.4.$$

Note that

$$\begin{aligned}\hat{r}_1(2) &= E[r_3|F_1] \\ &= E[I_2 r_2 + a_3|F_1] \\ &= E[I_2(0.4 + a_2)|F_1] \\ &= 0.4E[I_2] + E[I_2 a_2|F_1].\end{aligned}$$

Note that $I_2 = 0.4$ if $a_2 = 1$ and $I_2 = 0.2$ if $a_2 = -1$. Hence, $E[I_2] = \frac{1}{2} * 0.4 + \frac{1}{2} * 0.2 = 0.3$. Note that $I_2 a_2 = 0.4$ if $a_2 = 1$ and $I_2 a_2 = -0.2$ if $a_2 = -1$. Hence, $E[I_2 a_2] = \frac{1}{2} * 0.4 * 1 + \frac{1}{2} * 0.2 * -1 = 0.1$. It follows that

$$\hat{r}_1(2) = 0.4 * 0.3 + 0.1 = 0.22.$$

Problem 6

The absolute value of the coefficient in front of x_{t-1} is smaller than 1. Hence, $\{x_t\}$ is a weakly stationary (ARMA) time series (compare p. 36-37 in Tsay). Hence,

$$\begin{aligned}E(x_t) &= E(0.3x_{t-1} + a_t + 0.5a_{t-1}) \\ &= E(0.3x_{t-1}) \\ &= 0.3E(x_t)\end{aligned}$$

so that $E(x_t) = 0$.

Set $b = 0.3$ and $d = 0.5$ and let B denote the backshift operator. Then, the time series can be expressed as

$$(1 - bB)x_t = (1 + dB)a_t. \tag{3}$$

To express $\{x_t\}$ as an MA process means that we want to write it on the form

$$x_t = \sum_{j=0}^{\infty} \psi_j B^j a_t \tag{4}$$

(and our mission is therefore to find constants $\psi_0, \psi_1, \psi_2, \ldots$ such that (4) holds). This means that

$$(1 - bB)x_t = (1 - bB) \sum_{j=0}^{\infty} \psi_j B^j a_t. \tag{5}$$

From (3) and (5) we obtain

$$\begin{aligned}1 + dB =&(1 - bB) \sum_{j=0}^{\infty} \psi_j B^j \\ =&(1 - bB)(\psi_0 + \psi_1 B + \psi_2 B^2 + \ldots) \\ =&\psi_0 + (\psi_1 - b\psi_0)B + (\psi_2 - b\psi_1)B^2 + (\psi_3 - b\psi_2)B^3 ...,\end{aligned}$$

which directly implies

$$
\begin{aligned}
1 &= \psi_0 \\
d &= \psi_1 - b\psi_0 \\
0 &= \psi_j - b\psi_{j-1} \quad \text{for } j \geq 2,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= d + b \\
\psi_j &= b\psi_{j-1} = b^{j-1}(d+b) \quad \text{for } j \geq 2.
\end{aligned}
$$

Plugging in the numbers (i.e. using $d + b = 0.8$) and simplifying a bit yields

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_j &= 0.3^{j-1} 0.8 \quad \text{for } j \geq 1.
\end{aligned}
$$

Using this in (4) yields

$$
x_t = a_t + 0.8 \sum_{j=1}^{\infty} 0.3^{j-1} B^j a_t
$$

and we have thus rewritten $\{x_t\}$ as an MA process (of infinite order).