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MA 5020 - Abstract Algebra
Final Exam

Oct 27, 2020

Instructions: Textbooks, notes and calculators are not allowed. Unless told otherwise, you may quote results that you learned during the class. When you do, state precisely the result that you are using. Be sure to justify your answers, and show clearly all steps of your solutions. In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

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1. (a) [2 pts] Suppose G is a finite group, and m, n are positive integers, where m divides n. Furthermore, suppose G has an element of order n. Prove that G has an element of order m.

   (b) [2 pts] Suppose $f\colon G \to H$ is a surjective homomorphism between finite groups. Suppose H has an element of order n. Prove that G has an element of order n.

   (c) [1 pt] Suppose $f\colon G \to H$ is a surjective homomorphism between finite groups. Suppose G has an element of order n. Does it follow that H has an element of order n? Prove or give a counterexample.

2. [5 pts] Suppose G is a simple group of order 168. How many elements of order 7 does G have?

3. (a) [3 pts] Let G be a finite group and $N \triangleleft G$ a normal subgroup. Suppose G acts on a set X in such a way that the induced action of N on X is *transitive*. This means that for any two elements $x, y \in X$, there exists an element $n \in N$ such that $nx = y$.

   Let $x \in X$ and let G_x be the stabilizer of x. Prove that $G = G_x N$.

   (b) [2 pts] Let G be a finite group. Suppose $N \triangleleft G$ is a normal subgroup, $P \subset N$ is a Sylow subgroup of N and $N_G(P)$ is the normalizer of P in G.

   Prove that $G = N_G(P)N$.

4. (a) [3 pts] Let G be a group. Suppose G has a normal subgroup N of index 4, such that the quotient group G/N is not cyclic.

   Prove that G has three distinct normal subgroups of index 2, say we call them A, B, and C, such that $G = A \cup B \cup C$.

   (b) [2 pts] Show that the group $S_3 \times S_3$ has a normal subgroup of index 4 such that the quotient group is not cyclic. Describe explicitly the three normal subgroups of part (a) in this case.

5. Let R, S be not necessarily commutative rings with identity.

   (a) [2 pts] Suppose R is a division ring. Prove that the only (left, right or two-sided) ideals of R are $\{0\}$ and R

   (b) [1 pt] Suppose that elements $a, b \in R$ satisfy $aba = 1$. Prove that $ab = ba$ and a is a unit.

   (c) [1 pt] Let $f\colon R \to S$ be a ring homomorphism. Show that if $f(1) \neq 1$ then $f(1)$ is a zero divisor, or zero.

   (d) [1 pt] Show that there is a non-zero ring homomorphism $f\colon \mathbb{Z}/3 \to \mathbb{Z}/6$.

6. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers.
   (a) [2 pts] Use Euclid's algorithm to find a greatest common divisor of $9 + 3i$ and 5 with the property that its real part and complex part are positive.
   (b) [1 pt] Let $\gcd(9 + 3i, 5)$ be the answer that you found in part (a). Find $a, b \in \mathbb{Z}[i]$ such that $a(9 + 3i) + 5b = \gcd(9 + 3i, 5)$.
   (c) [2 pts] Prove that there is an isomorphism of rings

$$\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2 + 1).$$