MMS024 Logic — Exam 2020-08-21 — Solutions

1 (a) Derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{P_1\to\neg P_2 \quad [P_1]^2}{\neg P_2}{\scriptstyle \to E} \quad [P_2]^1}{\bot}{\scriptstyle \to E}}{\neg P_1}{\scriptstyle \to I_2}}{P_2\to\neg P_1}{\scriptstyle \to I_1}$$

(b) Derivable:

$$\dfrac{P_1\vee(P_2\wedge P_3) \quad \dfrac{\dfrac{[P_1]^1}{P_1\vee P_2}{\scriptstyle \vee I} \quad \dfrac{[P_1]^1}{P_1\vee P_3}{\scriptstyle \vee I}}{(P_1\vee P_2)\wedge(P_1\vee P_3)}{\scriptstyle \wedge I} \quad \dfrac{\dfrac{\dfrac{[P_2\wedge P_3]^1}{P_2}{\scriptstyle \wedge E}}{P_1\vee P_2}{\scriptstyle \vee I} \quad \dfrac{\dfrac{[P_2\wedge P_3]^1}{P_3}{\scriptstyle \wedge E}}{P_1\vee P_3}{\scriptstyle \vee I}}{(P_1\vee P_2)\wedge(P_1\vee P_3)}{\scriptstyle \wedge I}}{(P_1\vee P_2)\wedge(P_1\vee P_3)}{\scriptstyle \vee E_1}$$

2 (a) $FV(\exists x_2\ f_1(x_1) \doteq x_2) = \{x_1\}$

(b) $FV((\forall x_3\ P_{10}(x_7)) \to (\exists x_7\ P_{10}(x_3))) = \{x_3, x_7\}$

3.

	P_1	P_2	P_3
V_1	0	0	1
V_2	1	1	1
V_3	1	0	1

(a) "$P_1 \leftrightarrow P_2$" holds in V_1 and V_2, but not in V_3.

(b) "P_3" holds in V_1, V_2, V_3, but is not a tautology.

4. (a) "$\top$" and "$\top \to \top$" are distinct but logically equivalent formulas.

(b) Suppose $\varphi \simeq \psi$.

Then in any model V, if φ holds, so does ψ,

i.e. $\varphi \vDash \psi$.

So by <u>completeness</u>, $\varphi \vdash \psi$.

5.

$$\dfrac{\dfrac{\dfrac{[\exists x_1\, \neg\varphi]^1 \qquad \dfrac{[\neg\varphi]^3 \qquad \dfrac{[\forall x_1\, \varphi]^2}{\varphi}\,\forall E}{\bot}\,{\to}E}{\bot}\,\exists E_3}{\neg(\forall x_1\, \varphi)}\,{\to}I_2}{(\exists x_1\, \neg\varphi) \to \neg(\forall x_1\, \varphi)}\,{\to}I_1$$

with the substitution $\varphi[x_1/x_1] = \varphi$

6. (a) $\mathcal{N}, v \models P_1(x_1, x_2) \to P_1(x_2, x_1)$

$\iff$ if $0<0$, then $0<0$

which is <u>true</u>,

so $[\![P_1(x_1, x_2) \to P_1(x_2, x_1)]\!]^{\mathcal{N}, v} = 1$.

(b) $\mathcal{N}, v \models \forall x_1 (P_1(x_1, x_2) \to P_1(x_2, x_1))$

$\iff$ for all $n \in \mathbb{N}$, if $n<0$ then $0<n$

which is again <u>true</u> (since "$n<0$" is always false)

so $[\![\forall x_1 (P_1(x_1, x_2) \to P_1(x_2, x_1))]\!]^{\mathcal{N}, v} = 1$.

7. $(\exists x_1 P_1(x_1)) \to (\forall x_1 P_1(x_1))$ is <u>not</u> a tautology,

since it fails in many models, e.g.

$\mathcal{A} = \langle \mathbb{N}; \{\text{even numbers}\}; \rangle$ is a countermodel,

since there exist even natural numbers,

but not all natural numbers are even.

8. In soundness, we prove by induction on derivations

"for every derivation $\mathcal{D}$,
in any interp. V,
if all assumptions of $\mathcal{D}$ hold,
then the conclusion of $\mathcal{D}$ holds." } "$\mathcal{D}$ is sound"

Case where $\mathcal{D}$ concludes with $\vee$E-alt:

$$\underbrace{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \varphi\vee\psi\end{array}\qquad\begin{array}{c}\mathcal{D}_2\\ \vdots\\ \varphi\to\sigma\end{array}\qquad\begin{array}{c}\mathcal{D}_3\\ \vdots\\ \psi\to\sigma\end{array}}{\sigma}\;\vee E\text{-alt}}_{\mathcal{D}}$$

IH: $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ all sound.

Let V be some interp. in which all assns of $\mathcal{D}$ hold. Then all assns of $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ hold in V (since none are discharged by $\vee$E-alt).

So by IH, their conclusions $\varphi\vee\psi$, $\varphi\to\sigma$, and $\psi\to\sigma$ all hold in V.

"$V \models \varphi \vee \psi$" means that either $V \models \varphi$ or $V \models \psi$.

If $V \models \varphi$, then since $V \models \varphi \to \sigma$, we get that $V \models \sigma$.

Similarly, if $V \models \psi$, then since $V \models \psi \to \sigma$, $V \models \sigma$.

Either way, $V \models \sigma$, as required. $\square$

9. Call these structures N_+, $N_\times$.

(a) "$\forall x_1, x_2, x_3, (f_1(x_1, x_2) \doteq f_1(x_1, x_3) \to x_2 \doteq x_3)$"

holds in N_+ (since addition of any fixed natural is injective)

but not $N_\times$ (since multiplication by 0 is not injective).

(b) "$\exists x_1 \; f_1(x_1, x_1) \doteq x_1$"

holds in N_+ and $N_\times$ (since $0+0=0$, $0 \cdot 0 = 0$)

but is not a tautology, since it fails in e.g. the structure $\langle \mathbb{N} \setminus \{0\} ; ; + \rangle$.

10. For all these, note that given a structure $\mathcal{A} = \langle A;\ ; f \rangle$, (6)

$$\mathcal{A} \models \varphi_{inj} \iff f \text{ is injective}$$

$$\mathcal{A} \models \varphi_{surj} \iff f \text{ is surjective}$$

$$\mathcal{A} \models \varphi_{invol} \iff f \text{ is an involution}$$
(i.e. $f \circ f = id_A$).

(a) Any involution is a bijection (since it is a two-sided inverse for itself);

so any model of φ_{invol} is also a model of φ_{inj};

i.e. $\varphi_{invol} \models \varphi_{inj}$.

So by completeness, $\varphi_{invol} \vdash \varphi_{inj}$.

(b) Take some injective but non-surjective function, e.g. successor, $S : \mathbb{N} \to \mathbb{N}$.

Then $\langle \mathbb{N}; ; S \rangle$ shows $\varphi_{inj} \nvDash \varphi_{surj}$,

so by soundness, $\varphi_{inj} \nvdash \varphi_{surj}$.

(c) Take some bijection that is not an involution, e.g. $f : \mathbb{Z} \to \mathbb{Z}$
$f(n) := n+1$.

Then $\langle \mathbb{Z}; ; f \rangle$ shows that

$$\varphi_{inj}, \varphi_{surj} \nvDash \varphi_{invol}$$

so again by soundness, $\varphi_{inj}, \varphi_{surj} \nvdash \varphi_{invol}$.

11. (a) An inconsistent theory proves <u>everything</u>: ⑧

if $\Gamma \vdash \bot$, then for every φ,

$$\dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{\varphi}\,\bot E$$

shows that $\Gamma \vdash \varphi$.

So Γ is complete, since for any φ, $\Gamma \vdash \varphi$ (and also $\Gamma \vdash \neg\varphi$)!

(b) The empty theory $\emptyset$ is not complete: neither P_1 nor $\neg P_1$ is derivable from $\emptyset$.

(We see this by soundness:

the interp. $P_1^v := 0$ shows that $\emptyset \not\models P_1$,

while $P_1^v := 1$ shows $\emptyset \not\models \neg P_1$.)

(c) Suppose Γ max. cons. Then for any φ, we will show either $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$.

9

If $\Gamma \cup \{\varphi\}$ is consistent, then by maximality, $\Gamma \cup \{\varphi\} = \Gamma$, i.e. $\varphi \in \Gamma$, and so $\Gamma \vdash \varphi$.

Otherwise, if $\Gamma \cup \{\varphi\}$ is inconsistent, let $\mathcal{D}$ be some deriv. of $\perp$ from $\Gamma \cup \{\varphi\}$.

Then
$$\dfrac{\mathcal{D}\left\{\begin{array}{c}[\varphi]\\ \vdots \\ \perp\end{array}\right.}{\neg\varphi}\ \neg I$$
shows that $\Gamma \vdash \neg\varphi$.