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Suggested solutions

Exam: Introduction to Finance Mathematics (MT5009), 2021-05-25

Problem 1

(a) We find $B(0,1) = e^{-y(0,1)} = 0.9139$, $B(0,2) = e^{-2y(0,2)} = 0.8187$ and $B(1,2) = B(0,2)/B(0,1) = 0.8958$ (compare p. 249 in Capinski & Zastawniak).

(b) The interest we search for is $f(0,1,2) = (2y(0,2) - y(0,1))/(2-1) = 0.11$ (compare p. 251 in Capinski & Zastawniak). At time 2 you will therefore have $100e^{0.11} = 111.6278$.

Problem 2

(a) You need to invest P today, where P is the present value of the infinite stream of payments:

$$\begin{aligned} P &= C\frac{1}{1+\frac{r}{4}} + C\frac{1+\frac{g}{4}}{(1+\frac{r}{4})^2} + C\frac{(1+\frac{g}{4})^2}{(1+\frac{r}{4})^3} + \dots \\ &= C\frac{1}{1+\frac{r}{4}}\sum_{t=0}^{\infty}\left(\frac{1+\frac{g}{4}}{1+\frac{r}{4}}\right)^t = C\frac{1}{1+\frac{r}{4}}\frac{1}{1-\frac{1+\frac{g}{4}}{1+\frac{r}{4}}} = \frac{C}{\frac{r}{4}-\frac{g}{4}} = \frac{4C}{r-g}. \end{aligned}$$

With $C = \$100$, $g = 0.02$, and $r = 0.05$ we obtain $P \approx \$13\ 333.33$.

(b) If you receive payments for n quarters, the present value of the future payments is

$$\begin{aligned} P_n &= C\frac{1}{1+\frac{r}{4}} + C\frac{1+\frac{g}{4}}{(1+\frac{r}{4})^2} + \dots + C\frac{(1+\frac{g}{4})^{n-1}}{(1+\frac{r}{4})^n} \\ &= C\frac{1}{1+\frac{r}{4}}\sum_{t=0}^{n-1}\left(\frac{1+\frac{g}{4}}{1+\frac{r}{4}}\right)^t = C\frac{1}{1+\frac{r}{4}}\frac{1-(\frac{1+\frac{g}{4}}{1+\frac{r}{4}})^n}{1-\frac{1+\frac{g}{4}}{1+\frac{r}{4}}} = 4C\frac{1-(\frac{1+\frac{g}{4}}{1+\frac{r}{4}})^n}{r-g}. \end{aligned}$$

We want to find n such that $P_n \leq 4000$, hence

$$1-\left(\frac{1+\frac{g}{4}}{1+\frac{r}{4}}\right)^n \leq \frac{4000(r-g)}{4C} = \frac{3}{10},$$

hence

$$n \leq \frac{\log\left(\frac{7}{10}\right)}{\log\left(\frac{1+\frac{g}{4}}{1+\frac{r}{4}}\right)},$$

from which we obtain $n \leq 47.972\dots$. Thus n will be equal to 47 quarters, which means that you will receive payments for 11.75 years.

Problem 3

(a) The solution is given in Capinski & Zastawniak p. 83, Theorem 3.33. With the notation of the book, the formula is

$$\mathbf{w}_M = \frac{(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}}{(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}\mathbf{u}^\top}$$

Standard calculations give

$$\mathbf{C}^{-1} \approx \begin{pmatrix} 59.87 & -12.72 & -28.95 \\ -12.72 & 14.91 & 1.75 \\ -28.95 & 1.75 & 31.58 \end{pmatrix}$$

and

$$\mathbf{w}_M \approx \Big(-1.651, 0.636, 2.016 \Big).$$

(b) Since the market consists of three stocks and a risk-free asset, the portfolio with the largest expected return given a specific standard deviation can be found on the capital market line. Hence we invest x dollars in the risk-free asset and $(1000-x)$ dollars in the market portfolio. Since the risk-free return has no risk, the risk of this portfolio is

$$\sigma_P = \sqrt{(1000-x)^2\sigma_M^2} = |1000-x|\sigma_M,$$

and

$$\sigma_M = \sqrt{\mathbf{w}_M \mathbf{C} \mathbf{w}_M^\top}.$$

Standard calculations lead to

$$\sigma_M = 0.401\ldots$$

The requirement $\sigma_P \leq 100$ leads to

$$|1000-x| \leq \frac{100}{\sigma_M}.$$

Since we want to maximise the return, we want to invest as much as possible in the market portfolio, hence $1000-x > 0$, and the inequality above becomes an equality, i.e.

$$1000 - x = \frac{100}{\sigma_M} = 249.196\ldots$$

Hence we should invest $(1000-x) \approx \$249.20$ in the market portfolio, and $x \approx \$750.80$ in the risk-free asset.

Problem 4

What we are asked to do is to find the optimal hedge ratio, see around p 106-107 in Capinski & Zastawniak. If you enter into N short futures then your total position at t is

$$10S(t) - N(f(t,T) - f(0,T))$$

where first term is your wealth in the shares and the second term is the payment you make due to marking-to-market in the futures position. Hence, we should minimize

$$V(10S(t) - N(f(t,T) - f(0,T)).$$

Using that $f(0,T)$ is known at time 0 and that $f(t,T) = S(t)e^{r(T-t)}$ (see p. 101) we note that what we should minimize can be rewritten

$$\begin{aligned}
&V(10S(t) - N(f(t,T) - f(0,T)) \\
&= 10^2 V(S(t)) + N^2 V(f(t,T)) - 2 \cdot 10 N Cov(f(t,T), S(t)) \\
&= 100V(S(t)) + N^2 V(S(t)e^{r(T-t)}) - 20NCov(S(t)e^{r(T-t)}, S(t)) \\
&= 100V(S(t)) + N^2 e^{2r(T-t)} V(S(t)) - 20Ne^{r(T-t)} Cov(S(t), S(t)) \\
&= 100V(S(t)) + N^2 e^{2r(T-t)} V(S(t)) - 20Ne^{r(T-t)} V(S(t)).
\end{aligned}$$

Taking the derivative with respect to N and setting it to zero yields

$$N^* = \frac{10}{e^{r(T-t)}}$$

which is therefore (since the function we want to minimize is a quadratic function in N) the optimal number of contracts we should short.

Using the above we see that the corresponding minimal risk is

$$\begin{aligned}
&100V(S(t)) + (N^*)^2 e^{2r(T-t)} V(S(t)) - 20N^* e^{r(T-t)} V(S(t)) \\
&= 100V(S(t)) + (\frac{10}{e^{r(T-t)}})^2 e^{2r(T-t)} V(S(t)) - 20\frac{10}{e^{r(T-t)}} e^{r(T-t)} V(S(t)) \\
&= 100V(S(t)) + 100V(S(t)) - 200V(S(t)) \\
&= 0.
\end{aligned}$$

Problem 5

Recall that the time 0 expected value of a Wiener process is zero, for each t. Hence, we find (see around p 214 in Capinski & Zastawniak)

$$\begin{aligned}
V(0) &= e^{-rT} E_*[G(S(T))] \\
&= e^{-rT} E_* \left[ln \left[S(0) e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_*(T)} \right] \right] = \\
&= e^{-rT} E_* \left[ln\left[S(0)\right] + \left(r - \frac{1}{2}\sigma^2 \right) T + \sigma W_*(T) \right] = \\
&= e^{-rT} \left[ln\left[S(0)\right] + \left(r - \frac{1}{2}\sigma^2 \right) T \right].
\end{aligned}$$

Noting that $\sigma W_*(T) \in N(0, \sigma^2 T)$ and recalling from basic probability the moment generating function for the normal distribution, we find

$$\begin{aligned} V(0) &= e^{-rT} E_*[\tilde{G}(S(T))] \\ &= e^{-rT} E_* \left[S^n(0) e^{n(r-\frac{1}{2}\sigma^2)T + n\sigma W_*(T)} \right] \\ &= S^n(0) e^{((n-1)r - \frac{n}{2}\sigma^2)T} E_* \left[e^{n\sigma W_*(T)} \right] \\ &= S^n(0) e^{((n-1)r - \frac{n}{2}\sigma^2)T} e^{\frac{1}{2}\sigma^2 T n^2} \\ &= S^n(0) e^{(n-1)rT + \frac{1}{2}\sigma^2 (n^2 - n)T}. \end{aligned}$$