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Suggested solutions

Exam: Introduction to Finance Mathematics (MT5009), 2021-08-19

Problem 1

(A) The present value is

$$P = \sum_{t=0}^{10\cdot 12-1} \frac{C}{(1+r/12)^{t+35\cdot 12}} = \frac{C}{(1+r/12)^{35\cdot 12}} \sum_{t=0}^{119} (1+r/12)^{-t}$$
$$= \frac{C}{(1+r/12)^{420}} \frac{1-(1+r/12)^{-120}}{1-(1+r/12)^{-1}}.$$

With $C = 10\ 000$ kr and $r = 0.03$, we obtain $P \approx 363\ 784$ kr.

(B) We need to find the amount D deposited monthly such that the present value of the deposits equals the present value in (A). The present value of the deposits is

$$\sum_{t=0}^{35\cdot 12-1} \frac{D}{(1+r/12)^t} = D\sum_{t=0}^{419}(1+r/12)^{-t} = D\frac{1-(1+r/12)^{-420}}{1-(1+r/12)^{-1}}.$$

Setting this expression equal to P from (a) yields

$$D = P\frac{1-(1+r/12)^{-1}}{1-(1+r/12)^{-420}} = 1396.532\ldots$$

Hence John needs to deposit 1397 kr each month.

Problem 2

See e.g. sections 6.1-6.2 in Capinski & Zastawniak. Using the information in the question we find that

$$p_* = \frac{R-D}{U-D} = -D,$$
$$S^{uu} = S(0)(1+U)^2 = (2+D)^2,$$
$$S^{ud} = S(0)(1+U)(1+D) = (2+D)(1+D)$$
$$S^{dd} = S(0)(1+D)^2 = (1+D)^2.$$

and that $D \in (-1, 0)$ and $U = 1 + D \in (0, 1)$. Using this (which yields e.g. $(S^{dd} - X)_+ = 0$), the risk-neutral valuation formula gives

$$\begin{aligned} C_E &= \frac{1}{(1+R)^2} E_* \left[(S(2) - X)_+ \right] \\ &= \left[p_*^2 (S^{uu} - X)_+ + 2p_*(1-p_*)(S^{ud} - X)_+ + (1-p_*)^2 (S^{dd} - X)_+ \right] \\ &= \left[D^2((2+D)^2 - 1) - 2D(1+D)((2+D)(1+D) - 1)_+ \right] \\ &= D \left[D((2+D)^2 - 1) - 2(1+D)((2+D)(1+D) - 1)_+ \right]. \end{aligned}$$

Plugging in $D = -0.5$ into the formula gives $C_E = 0.3125$.

Problem 3

(A) The solution to the variance-minimisation problem is given in Capinski & Zastawniak p. 73. With the notation of the book, the formula is

$$\mathbf{w}_{\mathrm{MVP}} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top}.$$

Standard calculations give

$$\mathbf{C}^{-1} \approx \begin{pmatrix} 43.07 & -41.29 & 23.09 \\ -41.29 & 100.10 & -47.55 \\ 23.09 & -47.55 & 34.16 \end{pmatrix}$$

and

$$\mathbf{w}_{\mathrm{MVP}} \approx \Big(0.543, 0.246, 0.212 \Big).$$

(B) The solution is given in Capinski & Zastawniak p. 75. With the notation of the book, the formula is

$$\mathbf{w}_V = \mathbf{a}\mu_V + \mathbf{b}.$$

We have

$$2\mathbf{w}_V = \lambda_1 \mathbf{m}\mathbf{C}^{-1} + \lambda_2 \mathbf{u}\mathbf{C}^{-1}$$

and

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 2M^{-1} \begin{bmatrix} \mu_V \\ 1 \end{bmatrix}$$

where

$$M = \begin{bmatrix} \mathbf{m}\mathbf{C}^{-1}\mathbf{m}^\top & \mathbf{u}\mathbf{C}^{-1}\mathbf{m}^\top \\ \mathbf{m}\mathbf{C}^{-1}\mathbf{u}^\top & \mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top \end{bmatrix}.$$

Let $M_{11} = \mathbf{m}\mathbf{C}^{-1}\mathbf{m}^\top$, $M_{12} = \mathbf{u}\mathbf{C}^{-1}\mathbf{m}^\top = \mathbf{m}\mathbf{C}^{-1}\mathbf{u}^\top$, and $M_{22} = \mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top$. Then

$$M^{-1} = \frac{1}{M_{11}M_{22} - M_{12}^2} \begin{bmatrix} M_{22} & -M_{12} \\ -M_{12} & M_{11} \end{bmatrix}.$$

This leads to

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \frac{2}{M_{11}M_{22}-M_{12}^2}\begin{bmatrix}M_{22} & -M_{12}\\ -M_{12} & M_{11}\end{bmatrix}\begin{bmatrix}\mu_V\\ 1\end{bmatrix}$$
$$= \frac{2}{M_{11}M_{22}-M_{12}^2}\begin{bmatrix}M_{22}\mu_V - M_{12}\\ M_{11}-M_{12}\mu_V\end{bmatrix},$$

hence

$$\mathbf{w}_V = \frac{1}{M_{11}M_{22}-M_{12}^2}\big((M_{22}\mu_V - M_{12})\mathbf{m} + (M_{11}-M_{12}\mu_V)\mathbf{u}\big)\mathbf{C}^{-1}$$
$$= \frac{1}{M_{11}M_{22}-M_{12}^2}\big(\mu_V(M_{22}\mathbf{m} - M_{12}\mathbf{u}) + M_{11}\mathbf{u} - M_{12}\mathbf{m}\big)\mathbf{C}^{-1},$$

which leads to

$$\mathbf{a} = \frac{(M_{22}\mathbf{m}-M_{12}\mathbf{u})\mathbf{C}^{-1}}{M_{11}M_{22}-M_{12}^2},$$
$$\mathbf{b} = \frac{(M_{11}\mathbf{u}-M_{12}\mathbf{m})\mathbf{C}^{-1}}{M_{11}M_{22}-M_{12}^2}.$$

Hence,

$$\mathbf{a} \approx \begin{bmatrix}4.878 & -12.398 & 7.520\end{bmatrix}, \quad \mathbf{b} \approx \begin{bmatrix}-0.268 & 2.307 & -1.039\end{bmatrix},$$

and the weights of the portfolio on the minimum variance line is given by

$$\mathbf{w}_V = \mathbf{a}\mu_V + \mathbf{b},$$

with the numerical vectors $\mathbf{a}$ and $\mathbf{b}$ given above; and the weights of the portfolio on the minimum variance line with expected return 10% is

$$\mathbf{w}_V = 0.1\mathbf{a} + \mathbf{b} \approx \begin{bmatrix}0.220 & 1.067 & -0.287\end{bmatrix}.$$

Problem 4

(A) The derivative gives you the fixed amount 1 in case the price of the underlying exceeds 5 (the strike price) at maturity, and 0 otherwise (sometimes called a *cash-or-nothing option*).

(B) With $T = 1$, and the usual notation, we find (see around page 214 in Capinski & Zastawniak)

$$V(0) = e^{-rT}E_*\left(I_{\{S(T)\geq 5\}}\right)$$
$$= e^{-r}E_*\left(I_{\{S(0)e^{(r-\frac{1}{2}\sigma^2)+\sigma W_*(1)}\geq 5\}}\right)$$
$$= e^{-r}E_*\left(I_{\{W_*(1)\geq(ln(5/S(0))-(r-\frac{1}{2}\sigma^2))/\sigma\}}\right).$$

Hence, with some rewriting and basic probability we find

$$V(0) = e^{-r}E_*\left(I_{\{W_*(1)\geq(ln(5/S(0))-(r-\frac{1}{2}\sigma^2))/\sigma\}}\right)$$
$$= e^{-r}P_*\left(W_*(1) \geq (ln(5/S(0)) - (r-\frac{1}{2}\sigma^2))/\sigma\right)$$
$$= e^{-r}\left(1 - P_*\left(W_*(1) \leq (ln(5/S(0)) - (r-\frac{1}{2}\sigma^2))/\sigma\right)\right).$$

Recalling that $W_*(1)$ is a standard normal random variable and denoting the corresponding distribution function by N, we find the pricing formula

$$V(0) = e^{-r}\left(1 - N\left(\frac{ln(5/S(0)) - (r - \frac{1}{2}\sigma^2)}{\sigma}\right)\right),$$

or equivalently

$$V(0) = e^{-r} N\left(\frac{ln(S(0)/5) + (r - \frac{1}{2}\sigma^2)}{\sigma}\right).$$

Problem 5

(A) Plugging in the parameters into the Black-Scholes formula gives $C_E(0) = 54.78$ (Capinski & Zastawniak p. 215).

(B) See around Capinski & Zastawniak p. 227 for the VaR part of the question: Note that the share price at $T = 1$ exceeds 70.93 with probability 0.95 i.e.

$$P(S(1) > 70.93) = 0.95.$$

To see this let Z be a standard normal random variable and solve the equation

$$\begin{aligned} &P(S(1) > y) = 0.95 \\ \Leftrightarrow\ &P(S(0)e^{\mu+\sigma Z} > y) = 0.95 \\ \Leftrightarrow\ &P(Z < \frac{ln(y/S(0)) - \mu}{\sigma}) = 0.05 \\ \Leftrightarrow\ &y = S(0)e^{N^{-1}(0.05)\sigma+\mu} \end{aligned}$$

which yields $y = 70.93$ (recall that $N^{-1}(0.05) \approx -1.645$). Hence, the option will end up being in the money (i.e. $S(1) > X = 50$) with a probability exceeding 0.95[1]; i.e. the worst thing that can, at the 95% confidence level, happen is that our payoff is $70.93 - 50 = 20.93$; which corresponds to a loss of $54.78e^{0.1} - 20.93 = 39.61$, i.e. at the 95 % confidence level we have $VaR = 39.61$ (note that $54.78e^{0.1}$ is the payoff we would have gotten if we instead of buying the option would have invested the same amount in the risk-free rate).

Some further hints for the solution is to note that we aim at solving the equation

$$P(C_E(0)e^r - C_E(1) < VaR) = 0.95$$

which (using the observations above and plugging in numbers) corresponds to

$$\begin{aligned} &P(C_E(0)e^r - (S(1) - X) < VaR) = 0.95 \\ \Leftrightarrow\ &P(S(1) > C_E(0)e^r + X - VaR) = 0.95 \\ \Leftrightarrow\ &C_E(0)e^r + X - VaR = 70.93 \\ \Leftrightarrow\ &VaR = C_E(0)e^r + X - 70.93 = 39.61. \end{aligned}$$

[1]Note that a complete solution must include an observation of this type.