STOCKHOLM UNIVERSITY
Department of Mathematics
Examiner: Lionel Lang

Examination in
Mathematics for Economic and Statistical Analysis
Master Program, 7.5 ECTS
30th September 2020

Time: 13:00-18:00
Instructions:
- During the exam you CAN NOT use any textbook, class notes, or any other supporting material.
- Non-graphical calculators will be provided for the exam by the department. Other calculators MAY NOT be used.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language when appropriate, not just mathematical symbols.
- Write clearly and legibly.
- Mark clearly where is your final answer by putting $\boxed{\text{A BOX}}$ around it.

Grades: There are 6 problems. Each solved problem is awarded by up to 5 points. At least 15 points are necessary for the grade E. The problems are not ordered according to the difficulty.

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1. Let A be the matrix
$$A = \begin{pmatrix} 1 & -1 & 3 \\ 1 & k+2 & k+6 \\ -1 & 2 & k-3 \end{pmatrix}.$$
depending on the parameter $k \in \mathbb{R}$.
   (a) Compute the determinant $|A|$ as a function of k. $(2p)$
   (b) Determine the values of k for which the matrix A is invertible. $(1p)$
   (c) Solve the system of linear equations
$$\begin{pmatrix} 1 & -1 & 3 \\ 1 & 1 & 5 \\ -1 & 2 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix}$$
in the variables x, y and z. (Observe that the coefficient matrix is A for $k = -1$) $(2p)$

2. Consider the function $f(x, y) = e^{xy-x-y}$ defined on the compact set
$$D = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x \geqslant 0,\ y \geqslant 0,\ y \leqslant 4 - x \right\}.$$
   (a) Draw the set D and determine the boundary ∂D (to be expressed mathematically, as a subset or a union of subsets of $\mathbb{R}^2$). $(1p)$
   (b) Determine the critical points of f in the interior of D and compute the value of f at those points. $(2p)$
   (c) Determine the maximal and the minimal values of f on D. $(2p)$

3. Compute the primitives

$$\text{(a)} \quad \int \frac{6x^3 + 3x^2 - 2x + 5}{2x+1}\, dx, \quad (2p) \qquad\qquad \text{(b)} \quad \int 6x(x^2-1)^2 \ln(x^2-1)\, dx. \quad (3p)$$

4. Let $f(x, y) = e^{x^2 - y^2}$ and $g(x, y) = (x-1)^2 + y^2$. Our goal is to optimize the function f when the variables x and y are submitted to the constraint $g(x, y) = 4$.

   (a) Compute the gradients $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ and $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$. $\quad (2p)$

   (b) Solve in x and y the following equation

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = 0. \quad (2p)$$

   (c) Find the extremal values of f when x and y satisfy $g(x, y) = 4$. $\quad (1p)$

5. Compute the limits

$$\text{(a)} \quad \lim_{x \to +\infty} \frac{6x^2 - x}{2x+1} - \frac{3x^2 + x}{x-2}, \quad (2p) \qquad\qquad \text{(b)} \quad \lim_{x \to 0} \frac{x(e^x - 1)}{\sqrt{1-2x} - 1 + x}. \quad (3p)$$

6. Suppose y is defined implicitly as a function of x by $x^2 = y^3 + y + 2$.

   (a) Compute the derivative $\dfrac{dy}{dx}$ (express it as a function of x and $y(x)$). $\quad (3p)$

   (b) Find the only value $x = x_0$ satisfying $x > 0$ and $y(x) = 1$. $\quad (1p)$

   (c) Compute the linear approximation of $y(x)$ at the point $(x_0, 1)$. $\quad (1p)$

GOOD LUCK!