STOCKHOLM UNIVERSITY
Department of Mathematics
Examiner: Lionel Lang

Examination in
Mathematics for Economic and Statistical Analysis
Master Program, 7.5 ECTS
28th October 2020

Time: 13:00-18:00
Instructions:
- During the exam you CAN NOT use any textbook, class notes, or any other supporting material.
- Non-graphical calculators will be provided for the exam by the department. Other calculators MAY NOT be used.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language when appropriate, not just mathematical symbols.
- Write clearly and legibly.
- Mark clearly where is your final answer by putting $\boxed{\text{A BOX}}$ around it.

Grades: There are 6 problems. Each solved problem is awarded by up to 5 points. At least 15 points are necessary for the grade E. The problems are not ordered according to the difficulty.

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1. Let A be the matrix
$$A = \begin{pmatrix} 2 & 5 & -1 \\ -4 & k-11 & 5 \\ -2 & -5 & k \end{pmatrix}.$$
depending on the parameter $k \in \mathbb{R}$.
   (a) Compute the determinant $|A|$ as a function of k. (2p)
   (b) Determine the values of k for which the matrix A is invertible. (1p)
   (c) Solve the system of linear equations
$$\begin{pmatrix} 2 & 5 & -1 \\ -4 & -11 & 5 \\ -2 & -5 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$$
   in the variables x, y and z. (2p)

2. Consider the function $f(x, y) = \ln(1 + x^2 + y^2)$ defined on the compact set
$$D = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 - 1 \leqslant y \leqslant 1 - x\right\}.$$
   (a) Draw the set D and determine the intersection of D with the line $\{y = 1 - x\}$. (1p)
   (b) Determine the critical points of f and compute the value of f at those points. (2p)
   (c) Determine the maximal and the minimal values of f on D. (2p)

3. Compute the integrals

   (a) $\displaystyle\int_1^{(\frac{e+1}{2})^2} \frac{3\sqrt{t}+5}{2\sqrt{t}-1}\, dt,$ (2p)   (b) $\displaystyle\int (2x^3 - 2x)\big(e^{x^2-1} - 1\big)\, dx.$ (3p)

4. Let $f(x,y) = \sqrt{1+x^2+y^2}$ and $g(x,y) = \left(\frac{x}{2}\right)^2 + (y-1)^2$. Our goal is to optimize the function f when the variables x and y are submitted to the constraint $g(x,y) = 1$.

   (a) Compute $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ and $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$. $(2p)$

   (b) Solve in x and y the following equation

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = 0. \quad (2p)$$

   (c) Find the extremal values of f when x and y satisfy $g(x,y) = 1$. $(1p)$

5. Compute the limits

$$\text{(a)} \quad \lim_{x\to 0} \frac{\ln(1+x^2)}{e^{3x}-1-3x}, \quad (3p) \qquad\qquad \text{(b)} \quad \lim_{x\to 1^-} \frac{3x+2}{\sqrt{x}-1} - \frac{1}{x-1}. \quad (2p)$$

6. Consider the function $f(x) = e^{1+x-x^2}$.

   (a) Compute the Taylor polynomial $T_2(x)$ of order 2 of $f(x)$ at $x = 0$. $(3p)$

   (b) Show that for any x that satisfies $|x| < 10^{-1}$, the following inequality is true

$$|f(x) - T_2(x)| \quad < \quad e^{1{,}09} \cdot 10^{-3}.$$

   (hint: use the inequality $|a+b+c+d| < |a|+|b|+|c|+|d|$.) $(2p)$

Formulas

The Taylor polynomial $T_n(x)$ of order n of the function $f(x)$ at $x = x_0$ is

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

and the remainder $R_{n+1}(x) := f(x) - T_n(x)$ is given by

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

for some number ξ between x and x_0.
The solutions of the equation $ax^2+bx+c=0$ are given by $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ if $b^2-4ac \geqslant 0$.

GOOD LUCK!