MATEMATISKA INSTITUTIONEN
STOCKHOLM UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Numerisk analys
Teoriprov
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Argue carefully. You are allowed to use intermediate results in the preceding problem(s) which you were not able to solve.

There are 6 problems and 16 points (p) each except the last problem with 20p.

Grades: A: $p \geq 90$; B: $80 \leq p \leq 89$; C: $70 \leq p \leq 79$; D: $60 \leq p \leq 69$; E: $50 \leq p \leq 59$;

(1) Let $A = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 3 \\ 3 \\ 1 \end{pmatrix}$. In computation of the solution to the equation $Ax = b$ we know that b is perturbed by a vector δb with $\|\delta b\|_\infty \leq 0.01$.

(a) Give an upper bound for $\|\delta x\|_\infty$, where δx is the associated perturbation in the computed solution.

(b) Compute the condition number $\kappa_\infty(A)$ and compare it with the quotient between $\|\delta x\|/\|x\|$ och $\|\delta b\|/\|b\|$. Is the upper bound obtained by perturbation analysis tight?

Here is the inverse of A: $A^{-1} = \begin{pmatrix} 25 & -41 & 10 & -6 \\ -41 & 68 & -17 & 10 \\ 10 & -17 & 5 & -3 \\ -6 & 10 & -3 & 2 \end{pmatrix}$.

(2) Assume that the function $f(x)$ is three times continuously differentiable and α is a zero of f but not a zero of its derivative.

(a) Show that the iteration

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, n = 1, 2, ...$$

can be obtained by applying Newton's method to the function $g(x) = \frac{f(x)}{\sqrt{|f'(x)|}}$.

(b) Argue that when the second derivative is very close to zero, the iteration is almost the same as the Newton's method iteration

(c) Show that, if $\{x_n\}$, $n = 0, 1, 2, ...$, generated by the above iteration converges in a neighborhood of α, then the convergence is cubic.

(3) Assume that the function f is sufficiently smooth. Let $x_i = x_0 + ih$ and $h > 0$

(a) Show that the formula

$$f(x_{\frac{1}{2}}) \approx \frac{1}{2}f(x_0) + \frac{1}{2}f(x_1) + \frac{1}{8}hf'(x_0) - \frac{1}{8}hf'(x_1)$$

is exact for all third degree polynomials.

(b) Derive an asymptotical (approximation) error estimate.

(c) Use the formula and error estimate to determine $f(x) = e^{1/2}$, $x_0 = 0, x_1 = 0.2$ using 6-decimals. (Note that $e^{0.2} = 1.221403$.)

(4) (a) What is the characteristic polynomial of the matrix

$$F = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\gamma_0 \\ 1 & 0 & \cdots & 0 & -\gamma_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\gamma_{n-1} \end{pmatrix}?$$

(b) Let $p(\lambda) = a_n\lambda^n + \cdots + a_0$ och $\gamma_i = a_i/a_n$, $i = 0, ..., n-1$ and $a_n \neq 0$. Show how Gersgorin's Theorem can be applied to obtain the statement that all the zeros of $p(\lambda)$, $\lambda_1, ..., \lambda_n$ satisfy

(i) $|\lambda_i| \leq \max\left\{\left|\frac{a_0}{a_n}\right|, \max_{1\leq k\leq n-1}\left(1 + \left|\frac{a_k}{a_n}\right|\right)\right\}$

(ii) $|\lambda_i| \leq \max\left\{1, \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right|\right\}$.

(c) Compare these two estimates for $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda - 1$.

(d) How would you solve polynomial equations, especially the polynomial has multiple zeros?

(5) Consider the initial value problem $y'(x) = f(x, y(x))$, $y(x_0) = y_0$.

(a) Derive both implicit and explicit Euler's methods for solving of this problem. Name, for each of them, at least one advantage and disadvantage, respectively.

(b) Determine the region where the methods are absolutely stable for $f(x, y) = ay$, where a is a (possibly complex) constant.

(c) When is implicit Euler's method preferable? Why?

You have finished the exam if your homework point $p_h \geq 15$ (i.e. p=20). Do (6a) if $p_h \in [10, 15)$ (i.e. p=10); do (6a) and (6b) if $p_h \in [5, 10)$ (i.e. p=5). Note that all your p_h will be added.

(6) (a) Let $y := \varphi(p, q) = -p + \sqrt{p^2 + q}$.

(i) Given the relative input errors ε_p ε_q, determine the relative output error of the result y.

(ii) Show that the problem is well conditioned for $p > 0$, $q > 0$.

(iii) Propose a numerically stable algorithm to compute y.

(b) Consider a symmetric $n \times n$ matrix A.

(i) Show that the eigenvalue problem is well-condtioned.

(ii) Assume further that A is symmetric positive definite tridiagonal. Propose an $O(n)$ running time algorithm to compute the Cholesky factor.

(iii) The finite difference method applied to the two-point boundary value problem: $\frac{d^2y}{dx^2} = 12x^2, 0 \leq x \leq 1$ with $y(0) = y(1) = 0$, using $x_j = 0 + (j-1)h$, $(j = 1, ..., J+1)$, results in a linear system of equations with the coefficient matrix A being symmetric tridiagonal. How do you solve this system of equations? Do you invert the matrix A? What types of linear solver is more suitable if J is very large? Write down at least one such numerical algorithm and the conditions under which the algorithm works.

(c) We can apply Newton-Raphson's method to find the positive solution of the equation $x^2 - c = 0$ to approximate $\sqrt{c}$ for $c > 0$. Write down the iteration x_n. Show that for all $0 < x_0 < \infty$, the sequence $\{x_n\}$ quadratically converges to $\sqrt{c}$.

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