MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Tentamensskrivning i
Mathematics Methods for Economist MM3001
2021-10-25

Instructions:

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- Use of calculators is permitted for performing calculations. The use of graphic or programmable features is NOT permitted.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear solutions even if the answer is correct.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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(1) (5pt) Compute the degree 2 Taylor polynomial of the function $f(x) = \ln(x^3-1)$ around the point $x_0 = 2$.
Solution: We have that $f(2) = \ln(7)$. On the other side $f'(x) = 3x^2\frac{1}{x^3-1}$, so $f'(2) = \frac{12}{7}$. Lastly

$$f''(x) = 6x\frac{1}{x^3-1} - 9x^4(x^3-1)^{-2},$$

so $f''(2) = \frac{12}{7} - \frac{36}{49} = \frac{48}{49}$. We apply the formula and we get

$$p_2(x) = \ln(7) + \frac{12}{7}(x-2) + \frac{24}{49}(x-2)^2.$$

(2) Study the function $f(x) = (x^2-9)e^{5x}$. You should determine the following:
(a) (3pt) places where the function is 0, local minima o maxima, where the function is concave or convex;
(b) (2pt) sketch the graph of f. You should say if the function has limit when $x \to \pm\infty$, and, in this case what is such limit. However it is not necessary to compute in details these limits.

Solution: As the exponential is always positive we have that $f(x) = 0$ if and only if $(x^2-9) = 0$, which in turn is satisfied if, and only if $x = \pm 3$. In order to compute local maxima and minima we derivate the function and set the derivative to 0. We get

$$f'(x) = 2xe^{5x} + 5(x^2-9)e^{5x} = e^{5x}(5x^2+2x-45),$$

which is 0 if, and only if $5x^2+2x-45 = 0$. We use the resolutive formula for degreee two equations and we get two roots, $x_1 = \frac{-1+\sqrt{226}}{5}$ and $x_1 = \frac{-1-\sqrt{226}}{5}$. As the coefficient of x^2 is positive we have that $5x^2+2x-45 = 0$ is positive outside the roots and negative inside. We get the following sign table

		$\frac{-1-\sqrt{229}}{5}$		$\frac{-1+\sqrt{229}}{5}$	
$f'(x)$	+	0	-	0	+
$f(x)$	$\nearrow$	-	$\searrow$	-	$\nearrow$

So x_2 is a local min-point and x_1 is a local max point.

To discuss the convexity of the graph we need to compute the second derivative

$$f''(x) = 5e^{5x}(5x^2 + 2x - 45) + e^{5x}(10x + 2) = e^{5x}(25x^2 + 20x - 223).$$

This again is 0 if and only if $25x^2+20x-223 = 0$. Again using the resolutive formula we get two roots

$$x_1 = \frac{-10 + \sqrt{5675}}{25}$$

and

$$x_1 = \frac{-10 - \sqrt{5675}}{25}.$$

Again the second derivative of f is going to be positive outside the roots and negative between them, so the function is going to be convex outside the roots and concave between them.

As we have learned that the "exponential wins always against a polynomial" we have that the $\lim_{x\to-\infty} f(x) = 0$,while $\lim_{x\to\infty} f(x) = +\infty$. One could get both limits by applying de l'Hopital 2 times to $\frac{x^2-9}{e^{-5x}}$.

(3) The expression

$$x + y^3 - 2y = y^5 - x^2$$

defines $y = y(x)$ as a function of x.

(a) (1pt) Verify that the graph $y = y(x)$ passes through the point $(1, 1)$.

(b) (2pt) Find $y'(1)$.

(c) (2pt) find the equation of the tangent line to the graph $y = y(x)$ in the point $(1, 1)$.

Solution: We have that $0 = 1 + 1^3 - 2 = 1^5 - 1^2$. So the given curve passes through the point $(1, 1)$. To find $y'(1)$ we have to derivate implicitly the function. We get

$$1 + 3y'y^2 - 2y' = 5y'y - 2x.$$

We set $x = 1$ and $y = 1$ and we get

$$1 + 3y' - 2y' = 5y' - 2.$$

We use algebra and solve for y'. We obtain

$$y'(1) = \frac{1}{4}.$$

The equation of the tangent to the curve in $(1, 1)$ is of the form $y = mx+k$ where $m = y'(1) = \frac{1}{4}$. In order to find k we have just to impose passage through $(1, 1)$ we get

$$1 = \frac{1}{4} + k,$$

thus we deduce that $k = \frac{3}{4}$and that the tangent line has equation

$$y = \frac{1}{4}x + \frac{3}{4}$$

(4) Compute the following integrals:

(a) (3pt) $\int \left(x^2e^{5x^3} + \frac{20}{(7x + 1000)^{21}} + 4x \right) dx,$

(b) (2pt) $\int_0^{e^3} x^2 \ln(x)dx$.

Solution

$$\int \left(x^2e^{5x^3} + \frac{20}{(7x+1000)^{21}} + 4x\right) dx = \int x^2e^{5x^3}dx + \int \frac{20}{(7x+1000)^{21}}dx + \int 4xdx$$

We compute the three summands independently:

$$\int 4xdx = 2x^2 + C_1.$$

To compute $\int x^2e^{5x^3}dx$ we set $u = 5x^3$ so that $du = 15x^2dx$. Using change of variables we get

$$\begin{aligned}\int x^2e^{5x^3}dx &= \frac{1}{15}\int e^u du \\ &= \frac{1}{15}e^{5x^3} + C_2\end{aligned}$$

For $\int \frac{20}{(7x+1000)^21}dx$ we use the change of variable $7x + 1000 = u$. The formula yields

$$\begin{aligned}\int \frac{20}{(7x+1000)^21}dx &= \frac{20}{7}\int u^{-21}du \\ &= \frac{20}{7}\frac{1}{-21+1}u^{-20} + C_3 \\ &= -\frac{1}{7}(7x+1000)^{-20} + C_3.\end{aligned}$$

Combining the results we get that the integral in a) is

$$\frac{1}{15}e^{5x^3} - \frac{1}{7}(7x+1000)^{-20} + 2x^2 + C$$

For the second integral we use integration by parts. Observe that ln is not defined in 0, so we have to take limits

$$\begin{aligned}\int_0^{e^3} (x^2\ln(x)dx &= \lim_{a\to 0^+}\int_a^{e^3} x^2\ln(x)dx \\ &= \lim_{a\to 0^+}\left(\left[\frac{1}{3}x^3\ln(x)\right]_a^{e^3} - \int_a^{e^3}\frac{1}{3}x^3\frac{1}{x}dx\right) \\ &= \frac{1}{3}e^9\ln(e^3) - \lim_{a\to 0^+} a^3\ln(a) - \lim_{a\to 0^+}\int_a^{e^3}\frac{1}{3}x^2dx\end{aligned}$$

Now we use that the "logarithm loses to any power" so that $\lim_{a\to 0^+} a^3\ln(a) = 0$ and we get

$$\int_0^{e^3} (x^2\ln(x)dx = e^9 - \lim_{a\to 0^+}\left[\frac{1}{9}x^3\right]_a^{e^3}$$

$$= e^9 - \frac{1}{9}e^9 = \frac{8}{9}e^9.$$

(5) Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & k-4 & -3 \\ -2 & k+2 & k-5 \end{pmatrix}$$

(a) (2 pt) Compute the determinant of A, $|A|$, as a function of k.
(b) (1 pt) Find all the values of k for which A is not invertible.
(c) (2 pt) Use Gaussian elimination to solve the linear system

$$\begin{cases} x & +4y & +3z & = & 1 \\ -x & -3y & -3z & = & -1 \\ -2x & +3y & -4z & = & 3 \end{cases}$$

Solution: We perform elementary operation onnthe rows of A to make the computations of the determinant easier:

$$\begin{vmatrix} 1 & 4 & 3 \\ -1 & k-4 & -3 \\ -2 & k+2 & k-5 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 3 \\ 0 & k & 0 \\ -2 & k+2 & k-5 \end{vmatrix}$$
$$= k(1(k-5) - (-2)3) = k(k+1)$$

We see immediately that $\det(A) = 0$ if and only if $k = 0$ or $k = -1$, so we deduce that A is invertible if $k \neq 0, -1$.
To solve the system we perform elemetnary row operations to the matrix

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ -1 & -3 & -3 & -1 \\ -2 & 3 & -4 & 3 \end{array}\right)$$

We add the firs row to the second row:

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ -1 & -3 & -3 & -1 \\ -2 & 3 & -4 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ -2 & 3 & -4 & 3 \end{array}\right)$$

We add twice the first row to the third row

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ -2 & 3 & -4 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 11 & 2 & 5 \end{array}\right)$$

We swap the second and third row:

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 11 & 2 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & 11 & 2 & 5 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

From the last row we deduce that $y = 0$. We plug in this in the second row and find that $2z = 5$. Therefore $z = \frac{5}{2}$. We plug both the solutions we found for y and z in the first row, and get that $x + 3 \cdot \frac{5}{2} = 1$. We deduce that $x = -\frac{13}{2}$.

(6) We define the area D which contains all the $(x, y) \in \mathbb{R}^2$ which satisfy the following conditions:

$$x \geq 0, \quad y \geq 0, \quad x + y \leq 3.$$

Let $f(x, y) = 4 + x^3 + y^3 + 3xy$.
(a) (1pt) Draw a sketch of D.
(b) (4pt) Find the maximum and minimum value of $f(x, y)$ in D.
Solution: To find the maximum and the minimum value of $f(x, y)$ we need first to see if there are stationary points in the in the interior of D. To this aim we compute the partial derivatives of order one and we set them to 0:

$$\frac{\partial}{\partial x} f(x, y) = 3x^2 + 3y = 0$$

$$\frac{\partial}{\partial y} f(x, y) = 3y^2 + 3x = 0$$

From the first equation we deduce that $y = -x^2$, we plug that into the second equation and we get that $x^4 + x = 0$. The polynomial $x^4 + x$ factors in the following way:

$$x^4 + x = x(x + 1)(x^2 - x + 1)$$

we deduce that we have two solutions $x = 0$ and $x = -1$, which give $y = 0$ and $y = -1$ respectively. Now we observe that neither the point $(-1, -1)$ nor $(0, 0)$ lie in the interior of D. So the maximum and minimum of f must be on the boundary.

On the three angles of the triangle D the functions take values $f(0, 0) = 4$, and $f(0, 3) = f(3, 0) = 31$. We need to find the extreme values of the functions along the sides.

<u>*Side* $x = 0$</u>. We use the Lagrange multiplier method and we get $\mathcal{L}(x, y) = 4 + x^3 + y^3 + 3xy + \lambda x$. We compute the partial derivatives of $\mathcal{L}$ with respect to x, y, and λ.

$$\frac{\partial}{\partial x} \mathcal{L}(x, y) = 3x^2 + 3y + \lambda = 0;$$

$$\frac{\partial}{\partial y} \mathcal{L}(x, y) = 3y^2 + 3x = 0;$$

$$\frac{\partial}{\partial \lambda} \mathcal{L}(x, y) = x = 0.$$

from the third equation we get $x = 0$, and from the second equation we get $y = 0$. As $(0, 0)$ is the corner we get no new candidate from this side.

Alternative solution: We restrict manually f to the line $x = 0$. We get $f(0, y) = 4 + y^3$ is a function of y of which we can find the critical points. $\frac{d}{dy} f(0, y) = 3y^2 = 0$ if, and only if $y = 0$. Again we end up with the point $(0, 0)$ which is a corner and has already been taken in account.

<u>*Side* $y = 0$</u>. We use the Lagrange multiplier method and we get $\mathcal{L}(x, y) = 4 + x^3 + y^3 + 3xy + \lambda y$. We compute the partial derivatives of $\mathcal{L}$ with respect to x, y, and λ.

$$\frac{\partial}{\partial x} \mathcal{L}(x, y) = 3x^2 + 3y = 0;$$

$$\frac{\partial}{\partial y} \mathcal{L}(x, y) = 3y^2 + 3x + \lambda = 0;$$

$$\frac{\partial}{\partial \lambda} \mathcal{L}(x, y) = y = 0.$$

from the third equation we get $y = 0$, and from the first equation we get $x = 0$. As $(0, 0)$ is the corner we get no new candidate from this side.

Alternative solution: We restrict manually f to the line $y = 0$. We get $f(x, 0) = 4 + x^3$ is a function of x of which we can find the critical points. $\frac{d}{dx} f(x, 0) = 3x^2 = 0$ if, and only if $x = 0$. Again we end up with the point $(0, 0)$ which is a corner and has already been taken in account.

<u>*Side* $x + y = 3$</u>. We use the Lagrange multiplier method and we get $\mathcal{L}(x, y) = 4 + x^3 + y^3 + 3xy + \lambda(x + y - 3)$. We compute the partial derivatives of $\mathcal{L}$ with respect to x, y, and λ.

$$\frac{\partial}{\partial x} \mathcal{L}(x, y) = 3x^2 + 3y + \lambda = 0;$$

$$\frac{\partial}{\partial y} \mathcal{L}(x, y) = 3y^2 + 3x + \lambda = 0;$$

$$\frac{\partial}{\partial\lambda}\mathcal{L}(x,y) = x + y - 3 = 0.$$

from the third equation we get $y = 3 - x$. We plug this in in the first and second equation and get

$$3x^2 + 9 - 3x + \lambda = 0$$

$$3x^2 - 18x + 27 + 3x + \lambda = 0$$

We deduce that

$$3x^2 + 9 - 3x = -\lambda = 3x^2 - 15x + 27$$

which is equivalent to

$$18 - 12x = 0$$

has solution in $x = \frac{3}{2}$ from which we deduce that $y = \frac{3}{2}$. We compute $f(\frac{3}{2}, \frac{3}{2}) = \frac{35}{2}$ which is both bigger than 4 and smaller than 31.
Alternative solution: We restrict manually f to the line $x + y = 3$. We get

$$\begin{aligned} f(x, 3-x) &= 4 + x^3 + (3-x)^3 + 3x(3-x) \\ &= 4 + x^3 + 27 - 27x + 9x^2 - x^3 + 9x - 3x^2 \\ &= 6x^2 - 18x + 31 \end{aligned}$$

is a function of x of which we can find the critical points. $\frac{d}{dx}f(x, 3-x) = 12x - 18$ which is 0 when $x = \frac{3}{2}$. We get again the point $(\frac{3}{2}, \frac{3}{2})$.
Conclusion: The minimum of the function in D is 4, takent at the point $(0,0)$. The maximum is 31, taken at the points $(0,3)$ and $(3,0)$

FORMULAS

- Taylor polynomial of degree n of $f(x)$ around the point $x_0 = a$:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

GOOD LUCK!!!

Texten på svenska

(1) (5pt) Bestäm Taylor polynomet av grad 2 till funktionen $f(x) = \ln(x^3 - 1)$ kring punkten $x_0 = 2$.

(2) Låt $f(x) = (x^2 - 9)e^{5x}$. Underök funktionen: följande ska finnas med:
 (a) (3pt) Nollställen, lokala max och minpunkter och var funktionen är konkav eller konvex;
 (b) (2p) skissa grafen av funktionen. Det ska framgå om funktionen har gränsvärden då $x \to \pm\infty$ och vilka dessa i så fall är, men gränsvärdena behöver inte motiveras utförligt.

(3) Bestäm följande integraler:
 (a) (3pt) $\int \left(x^2 e^{5x^3} + \dfrac{20}{(7x + 1000)^{21}} + 4x \right) dx,$
 (b) (2pt) $\int_0^{e^3} x^2 \ln(x) dx.$

(4) Uttrycket

$$x + y^3 - 2y = y^5 - x^2$$

definerar $y = y(x)$ som en funktion av x.
 (a) (1pt) Se att graphen av $y = y(x)$ går genom punkten $(1, 1)$.
 (b) (2pt) Bestäm $y'(1)$.
 (c) (2pt) Hitta ekvationen till tangenten av grafen $y = y(x)$ i punkten $(1, 1)$.

(5) Låt

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & k-4 & -3 \\ -2 & k+2 & k-5 \end{pmatrix}$$

 (a) (2 pt) Räkna determinanten till A, $|A|$ som en funktion av k.
 (b) (1 pt) Hitta alla värden k sådana att A inte är invertibar.
 (c) (2 pt) Använd Gausselimination för att lösa det linjär systemet nedanför

$$\begin{cases} x & +4y & +3z & = & 1 \\ -x & -3y & -3z & = & -1 \\ -2x & +3y & -4z & = & 3 \end{cases}$$

 (d) Vi definearar området D som innehåler alla $(x, y) \in \mathbb{R}^2$ som upfyller att:

$$x \geq 0, \quad y \geq 0, \quad x + y \leq 3.$$

 Låt $f(x, y) = 4 + x^3 + y^3 + 3xy$.
  (i) (1pt) Skissa området D.
  (ii) (4pt) Bestäm max och min värde av $f(x, y)$ i D.