Solutions exam Probability III

November, 2021

Problem 1
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let X be a random variable being defined on this space. Let $\mathcal{A}$ be a sub-σ-field of $\mathcal{F}$, generated by a finite partition $\mathcal{P} = \{A_1, A_2, \cdots, A_n\}$.
(**a**) Provide the definition of an $\mathcal{F}$-measurable random variable. (Just using Proposition 3 of the cheat-sheet is not enough). (4p)

Solution: Given a sample space $(\Omega, \mathcal{F})$, a $\mathcal{F}$-measurable random variable is a function $X : \Omega \to \mathbb{R}$ with the property that the set $\{\omega \in \Omega : X(\omega) \in B\}$ belongs to $\mathcal{F}$ for each Borel set $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$.

For part (b) and (c), let $\hat{X} = \mathbb{E}[X|\mathcal{A}]$.
(**b**) Show that if Y is an $\mathcal{A}$-measurable random variable, then
$\mathbb{E}[(X - \hat{X})Y] = 0$. (4p)

Solution: By Thm 4.8 (iii) from A and D:

$$\begin{aligned}\mathbb{E}[(X - \hat{X})Y] = \mathbb{E}[XY] - \mathbb{E}[\hat{X}Y] = \mathbb{E}[\mathbb{E}[XY|\mathcal{A}]] - \mathbb{E}[\hat{X}Y] \\ = \mathbb{E}[\mathbb{E}[X|\mathcal{A}]Y] - \mathbb{E}[\hat{X}Y]\end{aligned}$$

where we have used that $Y$is $\mathcal{A}$ measurable and therefore can be taken out of the conditional expectation. The final line can be rewritten as $\mathbb{E}[\hat{X}Y] - \mathbb{E}[\hat{X}Y]$, which obviously is equal to 0.

(**c**) Show that if Z is an $\mathcal{A}$-measurable random variable, then
$\mathbb{E}[(X - Z)^2] \geq \mathbb{E}[(X - \hat{X})^2]$. (4p)

Solution:

$$\mathbb{E}[(X-Z)^2] = \mathbb{E}[([X-\hat{X}]+[\hat{X}-Z])^2] = \mathbb{E}[[X-\hat{X}]^2]+2\mathbb{E}[([X-\hat{X}][\hat{X}-Z])]+\mathbb{E}[[\hat{X}-Z]^2].$$

We note that $\hat{X} - Z$ is $\mathcal{A}$ measurable and therefore by part b) the second summand is 0. The third summand is at least 0 because it is the expectation of a square. and the statement in the question follows.

Problem 2 or $\lambda > 0$, let X_λ be Poisson distributed with parameter λ. That is

$$\mathbb{P}(X_\lambda = k) = \frac{\lambda^k}{k!}e^{-\lambda} \qquad \text{for } k \in \{0, 1, 2, \cdots\}$$

and $\mathbb{P}(X_\lambda = k) = 0$ for $k \notin \{0, 1, 2, \cdots\}$.

a) Compute $\psi_\lambda(t) = \mathbb{E}[e^{tX_\lambda}]$, for $t \in \mathbb{R}$. That is, compute the moment generating function of X_λ? (2p)

Solution:

$$\psi_\lambda(t) = \mathbb{E}[e^{tX_\lambda}] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}e^{-\lambda}e^{tk} = \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}e^{-\lambda} = e^{\lambda e^t}e^{-\lambda} = e^{\lambda(e^t-1)}.$$

Define $Y_\lambda = (X_\lambda - \lambda)/\sqrt{\lambda}$.

b) Compute $\hat{\psi}_\lambda(t) = \mathbb{E}[e^{tY_\lambda}]$, for $t \in \mathbb{R}$. That is, compute the moment generating function of Y_λ? (2p)

Solution: Note $Y_\lambda = X_\lambda/\sqrt{\lambda} - \sqrt{\lambda}$ and by standard properties of moment generating functions we have

$$\hat{\psi}_\lambda(t) = \psi_\lambda(t/\sqrt{\lambda})e^{-\sqrt{\lambda}t} = e^{\lambda(e^{t/\sqrt{\lambda}}-1)}e^{-\sqrt{\lambda}t}.$$

Let Z be a standard normal distributed random variable. That is Z has density

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2} \qquad \text{for } z \in \mathbb{R}.$$

c) Compute $\psi_Z(t) = \mathbb{E}[e^{tZ}]$, for $t \in \mathbb{R}$. That is, compute the moment generating function of Z? (4p)

Solution:

$$\psi_Z(t) = \mathbb{E}[e^{tZ}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-z^2/2}e^{tz}dz = \frac{1}{\sqrt{2\pi}}e^{t^2/2}\int_{-\infty}^{\infty} e^{-(z-t)^2/2}dz = e^{t^2/2}$$

In part d) you may use without proof that the characteristic function of Y_λ is given by $\hat{\psi}_\lambda(it)$ and the characteristic function of Z is given by $\psi_Z(it)$, where $i = \sqrt{-1}$.

d) Show that Y_λ converges in distribution to Z as $\lambda \to \infty$. (4p)

Solution: From part b) and the hint it follows that the characteristic function of Y_λ is given by $e^{\lambda(e^{it/\sqrt{\lambda}}-1)-\sqrt{\lambda}it}$. The exponent can be Taylor expended as

$$-t^2/2 - it^3/(6\sqrt{\lambda}) + O(1/\lambda).$$

So the characteristic function of Y_λ converges to $e^{-t^2/2}$ as $\lambda \to \infty$, which by part c) and the hint is the characteristic function of a standard normal distribution. And the result follows because convergence of characteristic functions implies convergence in distribution.

Problem 3

Let $X_1, X_2, \cdots$ be a sequence of random variables and X another random variable, all defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

a) Let $g : [0, \infty) \mapsto [0, \infty)$ be a strictly increasing function. Show that

$$\mathbb{P}(|X| > \epsilon) \leq \frac{\mathbb{E}[g(|X|)]}{g(\epsilon)}.$$

(3p)

Solution: Let $A = \{\omega, |X(\omega)| > \epsilon\}$. For all $x > 0$ and $\epsilon \geq 0$, we have $g(x) \geq \mathbb{1}(g(x) > g(\epsilon))g(\epsilon)$. Further $x > \epsilon$ if and only if $g(x) > g(\epsilon)$ because $g(\cdot)$ is strictly increasing. Therefore,

$$\mathbb{E}(g(|X|)) \geq \mathbb{E}(g(\epsilon)\mathbb{1}(A)) = g(\epsilon)\mathbb{P}(A) = g(\epsilon)\mathbb{P}(|X| > \epsilon),$$

and the statement follows.

b) Let $g(x) = x/(1+x)$. Show that $X_n \xrightarrow{\mathbb{P}} 0$ if and only if $g(|X_n|)$ converges in expectation to 0. (6p)

Solution: Assume that $g(|X_n|) \xrightarrow{1} 0$, i.e. assume that $\mathbb{E}[g(|X_n|)] \to 0$. Note that $g(x) = 1 - 1/(1+x)$ is strictly increasing in x. Then for $\epsilon > 0$ by part a):

$$\mathbb{P}(|X_n| > \epsilon) \leq \frac{\mathbb{E}[g(|X_n|)]}{g(\epsilon)} = \frac{1+\epsilon}{\epsilon}\mathbb{E}\left[g(|X_n|)\right] \to 0,$$

by $g(|X_n|) \xrightarrow{1} 0$.

Then assume that $X_n \xrightarrow{\mathbb{P}} 0$, i.e. $\mathbb{P}(|X_n| > \epsilon) \to 0$ for all $\epsilon > 0$. Note that $g(x) \in [0, 1)$ and is increasing for $x \in [0, \infty)$. So, for all $\epsilon > 0$

$$\mathbb{E}\left(g(|X_n|)\right) = \mathbb{E}\left(g(|X_n|)\mathbb{1}(X_n \leq \epsilon)\right) + \mathbb{E}\left(g(|X_n|)\mathbb{1}(X_n > \epsilon)\right)$$

$$\leq \mathbb{E}\left(\frac{\epsilon}{1+\epsilon}\mathbb{1}(X_n \leq \epsilon)\right) + \mathbb{E}\left(1 \times \mathbb{1}(X_n > \epsilon)\right) = \frac{\epsilon}{1+\epsilon}\mathbb{P}(|X_n| \leq \epsilon) + \mathbb{P}(|X_n| > \epsilon)$$

$$= \frac{\epsilon}{1+\epsilon}(1 - \mathbb{P}(|X_n| > \epsilon)) + \mathbb{P}(|X_n| > \epsilon) \to \frac{\epsilon}{1+\epsilon}$$

as $n \to \infty$ by $X_n \xrightarrow{\mathbb{P}} 0$. Since ϵ can be chosen arbitrary small, $\mathbb{E}\left(g(|X_n|)\right) = \mathbb{E}\left(|g(|X_n|) - 0|\right) \to 0$ and the statement of the question follows.

c) Show that if X_n converges in expectation to X then X_n converges in probability to X. (3p)

Solution: Using part a) with $g(x) = x$ gives that for $\epsilon > 0$

$$\mathbb{P}(|X_n - X| > \epsilon) \leq \mathbb{E}[|X_n - X|]/\epsilon \to 0$$

by the convergence in expectation of X_n to X.

Problem 4

Consider a supercritical Galton-Watson Branching Process $\{Z_0, Z_1, Z_2, \cdots\}$ with $Z_0 = 1$. That is, let $\{X_{ij}\}_{i=0,1,2,\cdots;j=1,2,\cdots}$ be independent and identically distributed random variables with the same distribution as the non-negative integer valued random variable X. Define

$$Z_0 = 1 \qquad \text{and} \qquad Z_{k+1} = \sum_{j=1}^{Z_k} X_{kj} \qquad \text{for } k \geq 0.$$

Assume $\mathbb{P}(X \geq 1) = 1$, $\mathbb{E}[X] = m > 1$ and $Var[X] = \sigma^2 < \infty$.

a) Show that for all $n \geq 1$

$$\mathbb{E}[Z_n] = m^n \qquad \text{and} \qquad \mathbb{E}[(Z_n)^2] = m^2\mathbb{E}[(Z_{n-1})^2] + m^{n-1}\sigma^2.$$

Deduce from this (e.g. by induction) that for $n \geq 1$,

$$\mathbb{E}[(Z_n)^2] = m^{2n} + \sigma^2 \sum_{k=1}^{n} m^{n-k} m^{2(k-1)} = m^{2n} + \sigma^2 m^{n-1} \frac{m^n - 1}{m - 1}. \qquad \text{(4p)}$$

Solution: First,

$$\mathbb{E}[Z_n] = \mathbb{E}[\mathbb{E}[Z_n|Z_{n-1}]] = m\mathbb{E}[Z_{n-1}] = \cdots = m^n Z_0 = m^n.$$

Then,

$$\mathbb{E}[(Z_n)^2] = \mathbb{E}[\mathbb{E}[(Z_n)^2|Z_{n-1}]] = \mathbb{E}[\mathbb{E}[(\sum_{j=1}^{Z_{n-1}} X_{n-1,j})(\sum_{k=1}^{Z_{n-1}} X_{n-1,k})|Z_{n-1}]]$$

$$= \mathbb{E}[\sum_{j=1}^{Z_{n-1}} \mathbb{E}[(X_{n-1,j})^2|Z_{n-1}]] + 2\mathbb{E}[\mathbb{E}[\sum_{j=1}^{Z_{n-1}-1} \sum_{k=j+1}^{Z_{n-1}} X_{n-1,j}X_{n-1,k}|Z_{n-1}]]$$

$$= \mathbb{E}[Z_{n-1}](\sigma^2 + m^2) + \mathbb{E}[Z_{n-1}(Z_{n-1} - 1)]m^2 = m^{n-1}\sigma^2 + m^2\mathbb{E}[(Z_{n-1})^2].$$

Now assume that $\mathbb{E}[(Z_n)^2] = m^{2n} + \sigma^2 \sum_{k=1}^{n} m^{n-k} m^{2(k-1)}$, then we finish the proof by noting that $\mathbb{E}[(Z_1)^2] = m^2 + \sigma^2 = m^{2\cdot 1} + \sigma^2 m^{1-1} \frac{m^1-1}{m-1}$ and

$$\mathbb{E}[(Z_{n+1})^2] = m^n\sigma^2 + m^2\mathbb{E}[(Z_n)^2] = m^n\sigma^2 + m^2 \left(m^{2n} + \sigma^2 \sum_{k=1}^{n} m^{n-k} m^{2(k-1)} \right)$$

$$= m^n\sigma^2 + m^{2(n+1)} + \sigma^2 \sum_{k=1}^{n} m^{n-k} m^{2(k)} = m^n\sigma^2 + m^{2(n+1)} + \sigma^2 \sum_{\ell=2}^{n+1} m^{n+1-\ell} m^{2(\ell-1)}$$

$$= m^{2(n+1)} + \sigma^2 \sum_{\ell=1}^{n+1} m^{n+1-\ell} m^{2(\ell-1)}.$$

b) Show that $W_n = m^{-n} Z_n$ converges almost surely to a random variable W as $n \to \infty$. (4p)

Solution:

$$\mathbb{E}[W_{n+1} | Z_0, Z_1, \cdots Z_n] = m^{-(n+1)} \mathbb{E}[Z_{n+1} | Z_n] = m^{-(n+1)} m Z_n = m^{-n} Z_n = W_n.$$

Also $\mathbb{E}[|W_n|] = \mathbb{E}[W_n] = \mathbb{E}[W_0] = 1 < \infty$. So, $W_0, W_1, \cdots$ is a martingale, with respect to filtration generated by $Z_0, Z_1, \cdots$. Furthermore, we have

$$\mathbb{E}[(W_n)^2] = m^{-2n} \mathbb{E}[(Z_n)^2] = 1 + \sigma^2 \frac{1 - m^{-n}}{m^2 - m} \to 1 + \sigma^2 (m^2 - m)^{-1} < \infty.$$

Here we have used that $m > 1$. The Martingale convergence theorem now gives the statement of the question.

c) Show that as $n \to \infty$,

$$\frac{\sum_{i=1}^n Z_i}{\sum_{j=0}^{n-1} Z_j} - m = \frac{\sum_{i=1}^n (Z_i - m Z_{i-1})}{\sum_{j=0}^{n-1} Z_j} \overset{a.s.}{\to} 0.$$

(4p)

Hint: Note that by $\mathbb{P}(X \geq 1) = 1$ we have $\mathbb{P}(Z_{n+1} \geq Z_n) = 1$ for all $n \in \mathbb{N}$ and therefore $\sum_{i=1}^n Z_i \to \infty$ as $n \to \infty$.

Solution: Use Theorem 21 of cheat sheet, with

$$S_n = \sum_{i=1}^n (Z_i - m Z_{i-1}) \qquad \text{and} \qquad f(x) = \max(1, x).$$

First observe that S_n is a martingale because $\mathbb{E}[Z_i - m Z_{i-1}] = 0$ for all i and then using that $\mathbb{E}[(Z_i - m Z_{i-1})(Z_j - m Z_{j-1})] = 0$ for $i \neq j$ and therefore that $(Z_i - m Z_{i-1})$ and $(Z_j - m Z_{j-1})$ are uncorrelated we obtain for all n

$$\mathbb{E}[(S_n)^2] = \sum_{i=1}^n \mathbb{E}[(Z_i - m Z_{i-1})^2] = \sum_{i=1}^n \sigma^2 \mathbb{E}[Z_{i-1}] < \infty.$$

Note that

$$\langle S \rangle_n = \sum_{k=1}^n \mathbb{E}((S_k - S_{k-1})^2 | \mathcal{F}_{k-1}) = \sum_{k=1}^n Var(Z_k | Z_{k-1}) = \sigma^2 \sum_{k=1}^n Z_{k-1} \to \infty$$

and thus we have that

$$\frac{\sum_{i=1}^n (Z_i - m Z_{i-1})}{\sum_{j=0}^{n-1} Z_j} = S_n / f(\langle S \rangle_n)$$

and apply the second part theorem 21.

Problem 5

Let N be a strictly positive integer, $X_0 = 1$ and $X_1, X_2, \cdots$ be a sequence of dependent non-negative integer valued random variables and

$$\underline{\mathcal{F}} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \cdots\}$$

be a filtration generated by these random variables.
For $m, n \in \{0, 1, \cdots, N\}$ and for $k \in \{0, 1, 2, \cdots\}$, set

$$\mathbb{P}(X_{k+1} = n | \mathcal{F}_k) = \mathbb{P}(X_{k+1} = n | X_k)$$

and

$$\mathbb{P}(X_{k+1} = n | X_k = m) = \binom{N}{n} \left(\frac{m}{N}\right)^n \left(1 - \frac{m}{N}\right)^{N-n}$$

Define $Y_k = \left(\frac{N}{N-1}\right)^k X_k(N - X_k)$ and let $T = \min\{k \geq 1; Y_k = 0\}$.

a) Show that $Y_0, Y_1, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$. (4p)

Solution: Using the binomial theorem (in last line below):

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \sum_{j=0}^{N} \binom{N}{j} \left(\frac{X_n}{N}\right)^j \left(1 - \frac{X_n}{N}\right)^{N-j} \left(\frac{N}{N-1}\right)^{n+1} j(N-j)$$

$$= \sum_{j=1}^{N-1} N(N-1) \frac{(N-2)!}{(j-1)!(N-j-1)!} \left(\frac{X_n}{N}\right)^j \left(1 - \frac{X_n}{N}\right)^{N-j} \left(\frac{N}{N-1}\right)^{n+1}$$

$$= N(N-1) \left(\frac{X_n}{N}\right) \left(\frac{N - X_n}{N}\right) \left(\frac{N}{N-1}\right)^{n+1} \sum_{j=1}^{N-1} \binom{N-2}{j-1} \left(\frac{X_n}{N}\right)^{j-1} \left(1 - \frac{X_n}{N}\right)^{N-j-1}$$

$$= \left(\frac{N}{N-1}\right)^n X_n(N - X_n) = Y_n$$

Furthermore, $\mathbb{E}(|Y_n|) = \mathbb{E}(Y_n) = \mathbb{E}(Y_1) = Y_0 = N - 1 < \infty$ by the above computation.

b) Show that

$$\frac{4(N-1)}{N^2}(1-1/N)^n \leq \mathbb{P}(T > n) \leq (1-1/N)^n.$$

(8p)

Hint: Note that if $Y_n \neq 0$ then $Y_n \geq \left(\frac{N}{N-1}\right)^n (N-1)$ and $Y_n \leq \left(\frac{N}{N-1}\right)^n N^2/4$.
Solution: by $Y_T = 0$ and $Y_{n+1} = 0$ if $Y_n = 0$, we have

$$\begin{aligned} N-1 = \mathbb{E}(Y_n) &= \mathbb{E}(Y_n|T > n)\mathbb{P}(T > n) + \mathbb{E}(Y_n|T \leq n)\mathbb{P}(T \leq n) \\ &= \mathbb{E}(Y_n|T > n)\mathbb{P}(T > n) + \mathbb{E}(Y_T|T \leq n)\mathbb{P}(T \leq n) \\ &= \mathbb{E}(Y_n|T > n)\mathbb{P}(T > n). \end{aligned}$$

So,

$$\mathbb{P}(T > n) = (N-1)/\mathbb{E}(Y_n|T > n).$$

Using the hint we obtain:

$$\left(\frac{N}{N-1}\right)^n (N-1) \leq \mathbb{E}(Y_n|T > n) \leq \left(\frac{N}{N-1}\right)^n \frac{N^2}{4}.$$

The statement of the question follows immediately from the final two displays.