STOCKHOLMS UNIVERSITET	MT7001
MATEMATISKA INSTITUTIONEN	TENTAMEN
Avd. Matematisk statistik	October 25, 2021

Exam Probability III

October 25, 2021 kl. 9–14

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Permissible tools: pen, paper and attached "cheat-sheet"

5 problems. Maximum of 60 points

	A	B	C	D	E
Needed points	50	45	40	35	30

Partial answers might be worth points, even if you cannot finish an answer! You are allowed to use results from the "cheat sheet" without proof, unless the proof is explicitly asked for in the question. You may also use other results discussed in the lectures or in the course material, such as the Borel-Cantelli Lemma's. If you use such a result refer to it by stating the theorem you are using or by referring to its proper name (e.g. Fatou's lemma), and explicitly check whether the conditions of the theorem are satisfied.

Throughout the exam $\mathbb{N}$ is the set of strictly positive integers and all limits are for $n \to \infty$.

NOTE: The examiner expects that subproblems marked with a $*$ might require more thought than other exercises and may be left for the end, if time permits

Problem 1

(**a**) Let $\mathcal{A}$ be a collection of subsets of a sample space Ω. What is the definition of the σ-field generated by $\mathcal{A}$? (3p)

(**b**) Let A and B be two subsets of Ω, that satisfy $A \cap B \neq \emptyset$, $A \cup B \neq \Omega$, $A \cap B \neq A$ and $A \cap B \neq B$ (i.e. A and B are overlapping, do not fill up Ω and neither A nor B is fully contained in the other set). Provide a partition $\mathcal{P}$ of Ω, such that the σ-field generated by A and B is the same as the σ-field generated by $\mathcal{P}$. (4p)

Hint: Make a picture.

(**c**$*$) Let A and B be as is part b). Let $\mathcal{F}_A$ be a σ-field that contains A, but does not contain B and let $\mathcal{F}_B$ be a σ-field that contains B, but does not contain A. Show that the σ-field generated by A and B necessarily contains at least one element that is neither in $\mathcal{F}_A$ nor in $\mathcal{F}_B$. (5p)

Hint: Assume that all elements of the σ-field generated by A and B are in $\mathcal{F}_A$ or in $\mathcal{F}_B$ (or in both) and deduce a contradiction.

Problem 2

Let X be geometrically distributed with parameter $p \in (0,1)$, that is

$$\mathbb{P}(X = k) = p(1-p)^{k-1}, \qquad \text{for } k \in \mathbb{N}$$

and $\mathbb{P}(X = k) = 0$ if $k \notin \mathbb{N}$.

a) Show that the probability generating function $g(s) = \mathbb{E}[s^X]$ of X for $s \in [0,1]$ is given by

$$g(s) = \frac{ps}{1-(1-p)s}.$$

(2p)

For $i \in \mathbb{N}$ and $j \in \mathbb{N}$ let $X_{i,j}$ be independent and identically distributed random variables all distributed as X. Set $Z_1 = X$ and define inductively $Z_{n+1} = \sum_{j=1}^{Z_n} X_{n,j}$. Define $g_n = \mathbb{E}[s^{Z_n}]$ for $n \in \mathbb{N}$ and $s \in [0,1]$.

b) Show that $g_{n+1}(s) = g_n(g(s))$, and use that to prove that

$$g_n(s) = \frac{p^n s}{1-(1-p^n)s}, \qquad \text{for } s \in [0,1] \text{ and } n \in \mathbb{N}.$$

(4p)

Remark: In what follows you may extend the domain of $g_n(s)$ without further proof to $s \in [0, 1/(1-p^n))$. So,

$$g_n(s) = \mathbb{E}[s^{Z_n}] = \frac{p^n s}{1-(1-p^n)s} \qquad \text{for } s \in [0, 1/(1-p^n))$$

and that

$$\psi_n(t) = \mathbb{E}[e^{tZ_n}] = g_n(e^t) \qquad \text{for } t < -\log(1-p^n) = |\log(1-p^n)|.$$

c*) Show that $p^n Z_n$ converges to an exponentially distributed random variable with expectation 1. (6p)

Problem 3

Let $A_1, A_2, \cdots$ be independent events on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define

$$A := \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m.$$

That is, the A_i's happen infinitely often.

a) Prove (a special case of) the first Borel-Cantelli Lemma:

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$$

implies $\mathbb{P}(A) = 0$. (3p)

b) Prove the second Borel-Cantelli Lemma:

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \infty$$

implies $\mathbb{P}(A) = 1$. (3p)

c) Let $X_1, X_2, \cdots$ be independent random variables. Show that $X_n \stackrel{a.s.}{\to} 0$ if and only if $\sum_{i=1}^{\infty} \mathbb{P}(|X_i| > 1/k) < \infty$ for all $k \in \mathbb{N}$. (6p)

Hint: You may use without proof that

$$\{X_n \not\to 0\} = \cup_{k=1}^{\infty} \cap_{N=1}^{\infty} \cup_{n=N}^{\infty} \{(|X_i| > 1/k\}).$$

Problem 4

Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with $\mathbb{E}[X_1] = 1$, $\mathbb{P}(X_1 = 1) < 1$ and $\mathbb{P}(X_1 > \epsilon) = 1$ for some $\epsilon > 0$. For $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the σ-field generated by $X_1, X_2, \cdots, X_n$ and $\underline{\mathcal{F}}$ the corresponding filtration.

a) For $n \in \mathbb{N}$ define $Y_n = \prod_{k=1}^{n} X_k$. Show that $Y_1, Y_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$. (4p)

b) Show that $\frac{1}{n} \sum_{i=1}^{n} \log(X_i)$ converges almost surely to a non-positive constant. (4p)

Remark: It can be shown (and used without proof in part c, if needed) that $\frac{1}{n} \sum_{i=1}^{n} \log(X_i)$ *converges to a strictly negative constant.*

c) Show that Y_n converges almost surely to 0. (4p)

Problem 5

Let $X_1, X_2, \cdots$ be independent and identically distributed random variables with

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2.$$

Let $\underline{\mathcal{F}}$ be the filtration generated by those random variables. Define $S_0 = 0$ and $S_n = \sum_{k=1}^{n} X_k$ for $n \in \mathbb{N}$. Let $a, b \in \mathbb{N}$. Let

$$T_{-a} = \inf\{k \in \mathbb{N}; S_k = -a\} \qquad \text{and} \qquad T_b = \inf\{k \in \mathbb{N}; S_k = b\},$$

be the hitting times of respectively $-a$ and b. Define $T = \min(T_{-a}, T_b)$.

a) Compute $p = \mathbb{P}(T_{-a} = T)$. (4p)

b) Show that $Y_n = (S_n)^2 - n$ is an $\underline{\mathcal{F}}$-martingale and show that $\mathbb{E}[T] = ab$. (4p)

c*) Compute $\mathbb{E}[TS_T]$. (4p)

Hint: Find a suitable martingale. You might compute $\mathbb{E}[(S_{n+1})^3|\mathcal{F}_n]$, *to get inspiration on which martingale would be suitable.*

Good Luck!

Reminder

σ-algebras, probability measures and expectation

Definition 1 *The Borel σ-algebra on $\mathbb{R}$, is the smallest σ-algebra generated by the open subsets of $\mathbb{R}$. This definition can be extended to $\mathbb{R}^d$ for $d \geq 1$.*

Definition 2

$$\limsup_{n\to\infty} A_n := \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m$$

$$\liminf_{n\to\infty} A_n := \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m$$

Proposition 3 *A random variable X is $\mathcal{F}$-measurable if and only if $\{X \leq x\} := \{\omega \in \Omega : X(\omega) \leq x\}$ belongs to $\mathcal{F}$.*

Definition 4 *The distribution measure μ_X of the random variable X is the probability measure on $(\mathbb{R}, \mathcal{B})$ defined by $\mu_X(B) = \mathbb{P}(X \in B)$ for Borel sets $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra.*

Proposition 5 *If the σ-algebra $\mathcal{A}$ is generated by a finite partition $\mathcal{P}$. Then the function Y is $\mathcal{A}$ measurable if and only if Y is constant on each element of $\mathcal{P}$.*

Lemma 6 *If X, Y satisfy $\min(\mathbb{E}(X^+), \mathbb{E}(X^-)) < \infty$, then*
(i) $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$ *(linearity)*
(ii) $\mathbb{E}(X) \leq \mathbb{E}(Y)$ *if $X \leq Y$ a.s.* *(monotonicity)*

Definition 7 *Let $\mathcal{P} = \{A_1, \cdots, A_n\}$ be a finite partition, which generates the σ-algebra $\mathcal{A} \subset \mathcal{F}$, then $\mathbb{E}(X|\mathcal{A})(\omega) = \sum_{i=1}^{n} \mathbb{E}(X|A_i)\mathbb{1}(\omega \in A_i)$ for $\omega \in \Omega$*

Lemma 8 (Jensen's inequality) *We have $\mathbb{E}(\phi(X)) \geq \phi(\mathbb{E}(X))$ for convex functions ϕ.*

Characteristic functions

Definition 9 *the Characteristic function of a random variable X is the function $\varphi : \mathbb{R} \to \mathbb{C}$, defined by $\varphi_X(t) = \mathbb{E}(e^{itX}) = \mathbb{E}(\cos[tX]) + i\mathbb{E}(\sin[tx])$ where $i = \sqrt{-1}$.*

Properties of φ_X:

- $\varphi_X(0) = 1$
- $|\varphi_X(t)| \leq 1$
- $\varphi_X(-t) = \overline{\varphi_X(t)}$
- If $a, b \in \mathbb{R}$ and $Y = aX + b$ then $\varphi_Y(t) = e^{itb}\varphi_X(at)$
- If the random variables X and Y are independent, then $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$
- φ_X is real if and only if X and $-X$ have the same distribution, (X is symmetric)

Theorem 10 *Let X be a random variable with distribution function F and characteristic function φ. If F is continuous in both a and b, then*

$$F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itb} - e^{-ita}}{-it} \varphi(t)dt$$

special cases:

- If $\int_{\mathbb{R}} |\varphi(t)|dt < \infty$, then X has a continuous distribution with density

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\varphi(t)dt$$

- If the distribution of X is discrete, then

$$\mathbb{P}(X = x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx}\varphi(t)dt$$

Theorem 11 *Let $\varphi^{(k)}(\cdot)$ be the k-th complex derivative of φ.*

- *If $\varphi_X^{(k)}(0)$ exists then $\mathbb{E}(|X^k|) < \infty$ if k is even and $\mathbb{E}(|X^{k-1}|) < \infty$ if k is odd*
- *if $\mathbb{E}(|X^k|) < \infty$ then $\varphi_X(t) = \sum_{j=0}^{k} \frac{\mathbb{E}(X^j)}{j!}(it)^j + o(t^k)$, where $f(x) = o(x)$ if $f(x)/x \to 0$ for $x \to 0$*

Some useful results for convergence results

Chebychev's inequality: $\mathbb{P}(|X| > x) \leq \frac{\mathbb{E}(X^2)}{x^2}$

Markov inequality: $\mathbb{P}(|X| > x) \leq \frac{\mathbb{E}(|X|^r)}{x^r}$

Hölder's inequality: For $p, q > 1$ such that $1/p + 1/q = 1$ we have

$$\mathbb{E}(|XY|) \leq [\mathbb{E}(|X|^p)]^{1/p}[\mathbb{E}(|X|^q)]^{1/q}$$

Minkovski's inequality: For $r \geq 1$ we have

$$[\mathbb{E}(|X+Y|^r)]^{1/r} \leq [\mathbb{E}(|X|^r)]^{1/r} + [\mathbb{E}(|X|^r)]^{1/r}$$

Lemma 12 (Fatou's Lemma) *Let $X_1, X_2, \cdots$ be non-negative random variables, then $\mathbb{E}(\liminf X_n) \leq \liminf \mathbb{E}(X_n)$.*

Definition 13 (Tail events) *If $X_1, X_2, \cdots$ are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{H}_n = \sigma(X_{n+1}, X_{n+2}, \cdots)$ is the smallest σ-algebra in which all random variables $X_{n+1}, X_{n+2}, \cdots$ are measurable, then $\mathcal{H}_\infty := \cap_n \mathcal{H}_n$ is called the tail σ-algebra, and events contained in it are tail events.*

Theorem 14 (Kolmogorov's zero-one law) *If $X_1, X_2, \cdots$ are independent, then all tail events $H \subset \mathcal{H}_\infty$ satisfy either $\mathbb{P}(H) = 1$ or $\mathbb{P}(H) = 0$*

Definition 15 (Uniform integrability) *A sequence of r.v. $X_1, X_2, \cdots$ is uniformly integrable if*

$$\sup_{n \geq 1} \mathbb{E}(|X_n| \mathbb{1}(|X_n| > a)) \to 0 \qquad \text{as } a \to \infty$$

Theorem 16 *Let X and $X_1, X_2, \cdots$ be random variables such that $X_n \xrightarrow{\mathbb{P}} X$ then the following statements are equivalent*

1. *$X_1, X_2, \cdots$ is uniformly integrable*
2. *$\mathbb{E}(|X_n|) < \infty$ for all n, $\mathbb{E}(|X|) < \infty$ and $X_n \xrightarrow{1} X$*
3. *$\mathbb{E}(|X_n|) < \infty$ and $\mathbb{E}(|X_n|) \to \mathbb{E}(|X|) < \infty$*

Martingales

Some Properties of martingales: Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}} = (\mathcal{F}_0, \mathcal{F}_1, \cdots)$.

- $\mathbb{E}(S_{n+m}|\mathcal{F}_n) = S_n$
- $\mathbb{E}(S_n) = \mathbb{E}(S_1)$
- $\mathbb{E}((S_n)^2)$ is non decreasing

Theorem 17 (Doob decomposition) *A $\underline{\mathcal{F}}$-submartingale $Y_0, Y_1, \cdots$ with finite means may be expressed in the form $Y_n = M_n + S_n$, where $M_1, M_2, \cdots$ is a $\underline{\mathcal{F}}$-martingale and S_n is $\mathcal{F}_{n-1}$ measurable for all n. This decomposition is unique.*

Lemma 18 (Doob-Kolmogorov inequality) *If $S_1, S_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$, then for all $\epsilon > 0$ we have $\mathbb{P}\left(\max_{1\le k\le n} |S_k| \ge \epsilon\right) \le \epsilon^{-2}\mathbb{E}((S_n)^2)$.*

Theorem 19 (Martingale convergence theorem) *If $S_1, S_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$ and $\mathbb{E}((S_n)^2) \nearrow M < \infty$, then there exists a random variable S such that $S_n \stackrel{a.s.}{\to} S$.*

Definition 20 (Cauchy sequence) *A sequence of real numbers $x_1, x_2, \cdots$ is a Cauchy sequence if for all $\epsilon > 0$ there exists an N such that for all $n \ge m \ge N$, we have $|x_n - x_m| < \epsilon$.*
We know that a sequence is convergent if and only if it is a Cauchy sequence.

Theorem 21 *Let $S_0, S_1, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$ such that $S_0 = 0$ and $\mathbb{E}((S_n)^2) < \infty$ for all n. Define*

$$\langle S\rangle_n = \sum_{k=1}^{n} \mathbb{E}((S_k - S_{k-1})^2|\mathcal{F}_{k-1}) \qquad \textit{and} \qquad \langle S\rangle_\infty = \lim_{n\to\infty} \langle S\rangle_n.$$

Let $f \ge 1$ be a given increasing function satisfying $\int_0^\infty [f(x)]^{-2}dx < \infty$. Then,
(i) On $\{\omega : \langle S(\omega)\rangle_\infty < \infty\}$ $S_n \stackrel{a.s.}{\to} S$ for some random variable S
(ii) On $\{\omega : \langle S(\omega)\rangle_\infty = \infty\}$, $S_n/f(\langle S\rangle_n) \stackrel{a.s.}{\to} 0$

Theorem 22 (Strong Law of Large Numbers) *Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$ and define $S_0 = 0$ and $S_n = \sum_{k=1}^{n}(X_k - \mu)$ for $n \ge 1$. Then $\frac{S_n}{n} \stackrel{a.s.}{\to} 0$.*

Theorem 23 (Martingale Central Limit theorem) *$S_0, S_1, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$, with $S_0 = 0$ and $\mathbb{E}((S_n)^2) < \infty$ for all n. Assume that $n^{-1}\langle S\rangle_n \xrightarrow{\mathbb{P}} \sigma^2 > 0$ and for all $\epsilon > 0$*

$$\frac{1}{n}\sum_{k=1}^{n} \mathbb{E}((S_k - S_{k-1})^2 \mathbb{1}((S_k - S_{k-1})^2 > \epsilon n)) \to 0.$$

Then, $\frac{1}{\sqrt{n\sigma^2}} S_n \xrightarrow{d} \mathcal{N}(0,1)$

Theorem 24 (Optional stopping I) *Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$. If T is an a.s. bounded stopping time for $\underline{\mathcal{F}}$ (i.e. $\mathbb{P}(T \leq a) = 1$ for some $a \geq 0$), then $\mathbb{E}(S_T) = \mathbb{E}(S_1)$.*

Theorem 25 (Optional stopping II) *Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$ and T a stopping time for $\underline{\mathcal{F}}$. Then $\mathbb{E}(S_T) = \mathbb{E}(S_1)$, if the following conditions hold*

- $\mathbb{P}(T < \infty) = 1$,
- $\mathbb{E}(|S_T|) < \infty$,
- $\mathbb{E}(S_n \mathbb{1}(T > n)) \to 0$ *as* $n \to \infty$.

Theorem 26 (Optional Stopping III) *Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$ and T a stopping time for $\underline{\mathcal{F}}$. Then $\mathbb{E}(S_T) = \mathbb{E}(S_1)$, if the following conditions hold*

- $\mathbb{E}(T) < \infty$,
- $\mathbb{E}(|S_{n+1} - S_n||\mathcal{F}_n) \leq K$ *for all* $n < T$ *and some* $K > 0$

Wald's equation and identity: If $X_1, X_2, \cdots$ are i.i.d. random variables with $\mathbb{E}(X_1) = \mu < \infty$ and $S_n = \sum_{k=1}^{n} X_k$ and T is a stopping time satisfying $\mathbb{E}(T) < \infty$, then $\mathbb{E}(S_T) = \mu\mathbb{E}(T)$.
If in addition there exists a $h > 0$ such that $M(t) = \mathbb{E}(e^{tX_1}) < \infty$ for all $|t| < h$ and $M(t) > 1$ and $|S_n| < C$ for some constant $C > 0$ and all $n \leq T$, then $\mathbb{E}(e^{tS_T}[M(t)]^{-T}) = 1$.