Solutions exam Probability III

October, 2021

Problem 1

(**a**) Let $\mathcal{A}$ be a collection of subsets of a sample space Ω. What is the definition of the σ-field generated by $\mathcal{A}$? (3p)

Solution: The σ-field generated by $\mathcal{A}$ is the smallest σ-field that contains all elements of $\mathcal{A}$.

(**b**) Let A and B be two subsets of Ω, that satisfy $A \cap B \neq \emptyset$, $A \cup B \neq \Omega$, $A \cap B \neq A$ and $A \cap B \neq B$ (i.e. A and B are overlapping, do not fill up Ω and neither A nor B is fully contained in the other set). Provide a partition $\mathcal{P}$ of Ω, such that the σ-field generated by A and B is the same as the σ-field generated by $\mathcal{P}$. (4p)

Solution: The partition consists of

- $P_1 = A \cap B$,
- $P_2 = A \cap B^C$,
- $P_3 = A^C \cap B$ and
- $P_4 = A^C \cap B^C$.

This is a partition because $P_1 \cup P_2 = A$ and $P_3 \cup P_4 = A^C$. So, $\cup_{i=1}^4 P_i = \Omega$ and $(P_1 \cup P_2) \cap (P_3 \cup P_4) = \emptyset$. Finally $P_1, P_3 \subset B$ and $P_2, P_4 \subset B^C$. So, $P_1 \cap P_2 = \emptyset$ and $P_3 \cap P_4 = \emptyset$.

None of the elements of the partition is empty, because by assumption P_1, P_2 and P_3 are not empty, while $P_4 \neq \emptyset$, because

$$P_4^C = (A^C \cap B^C)^C = A \cup B \neq \Omega = \emptyset^C.$$

Since complements, intersections and unions of elements of a σ-field are also in the σ-field. All elements of the partition are in the smallest σ-field containing A and B. While $A = P_1 \cup P_2$ and $B = P_1 \cup P_3$ are in the smallest σ algebra generated by the partition.

(**c**) Let A and B be as is part b). Let $\mathcal{F}_A$ be a σ-field that contains A, but does not contain B and let $\mathcal{F}_B$ be a σ-field that contains B, but does not contain A. Show that the σ-field generated by A and B necessarily contains at least one element that is neither in $\mathcal{F}_A$ nor in $\mathcal{F}_B$. (5p)
Solution: Assume first that P_1, P_2, P_3 and P_4 are all in $\mathcal{F}_A \cup \mathcal{F}_B$ and that $P_1 = A \cap B \in \mathcal{F}_A$. (If $P_1 \notin \mathcal{F}_A$ Then the roles of A and B (and therefore P_2 and P_3) can be interchanged

- $P_3 = A^C \cap B \notin \mathcal{F}_A$, because $P_3 \in \mathcal{F}_A$ would (by definition of a σ-field and $P_1 \in \mathcal{F}_A$) imply $P_1 \cup P_3 \in \mathcal{F}_A$. However, $P_1 \cup P_3 = B \notin \mathcal{F}_A$ by assumption. So $P_3 \in \mathcal{F}_B$.

- Similarly, $P_3 \in \mathcal{F}_B$ implies $P_4 = A^C \cap B^C \notin \mathcal{F}_B$, because $P_4 \cup P_3 = A^C \notin \mathcal{F}_B$, because $A \notin \mathcal{F}_B$ by assumption. So $P_4 \in \mathcal{F}_A$.

- $P_4 \in \mathcal{F}_A$ implies $P_4 \cup A \in \mathcal{F}_A$. Since $P_4 \cup A = P_1 \cup P_2 \cup P_4 = (P_3)^C$ we have $P_3 \in \mathcal{F}_A$. Which contradicts the first bulletpoint.

Problem 2

Let X be geometrically distributed with parameter $p \in (0,1)$, that is

$$\mathbb{P}(X = k) = p(1-p)^{k-1}, \qquad \text{for } k \in \mathbb{N}$$

and $\mathbb{P}(X = k) = 0$ if $k \notin \mathbb{N}$.

a) Show that the probability generating function $g(s) = \mathbb{E}[s^X]$ of X for $s \in [0,1]$ is given by $g(s) = \frac{ps}{1-(1-p)s}$. (2p)

Solution:

$g(s) = \sum_{k=1}^{\infty} \mathbb{P}(X = k)s^k = \sum_{k=1}^{\infty} p(1-p)^{k-1}s^k = ps\sum_{k=1}^{\infty}[(1-p)s]^{k-1} = \frac{ps}{1-(1-p)s}$.

For $i \in \mathbb{N}$ and $j \in \mathbb{N}$ let $X_{i,j}$ be independent and identically distributed random variables all distributed as X. Set $Z_1 = X$ and define inductively $Z_{n+1} = \sum_{j=1}^{Z_n} X_{n,j}$. Define $g_n = \mathbb{E}[s^{Z_n}]$ for $n \in \mathbb{N}$ and $s \in [0,1]$.

b) Show that $g_{n+1}(s) = g_n(g(s))$, and use that to prove that

$$g_n(s) = \frac{p^n s}{1-(1-p^n)s}, \qquad \text{for } s \in [0,1] \text{ and } n \in \mathbb{N}.$$

(4p)

Solution:

$$g_{n+1}(s) = \mathbb{E}[s^{Z_{n+1}}] = \mathbb{E}[\mathbb{E}[s^{\sum_{j=1}^{Z_n} X_{n,j}}]|Z_n] = \mathbb{E}[\prod_{j=1}^{Z_n} \mathbb{E}[s^{X_{n,j}}|Z_n]],$$

where we have used independence of the $X_{n,j}$. Because all $X_{n,j}$ are distributed as X, we further obtain

$$g_{n+1}(s) = \mathbb{E}[\prod_{j=1}^{Z_n} \mathbb{E}[s^{X_{n,j}}|Z_n]] = \mathbb{E}[\prod_{j=1}^{Z_n} \mathbb{E}[s^X]] = \mathbb{E}[g(s)^{Z_n}] = g_n(g(s)).$$

We use induction to show $g_n(s) = \frac{p^n s}{1-(1-p^n)s}$. It is easy to see that $g_n(s) = \frac{p^n s}{1-(1-p^n)s}$ for $n = 1$ by a. If $g_n(s) = \frac{p^n s}{1-(1-p^n)s}$ then

$$\begin{aligned} g_{n+1}(s) = g_n(g(s)) = \frac{p^n g(s)}{1-(1-p^n)g(s)} &= \frac{p^n ps/[1-(1-p)s]}{1-(1-p^n)ps/[1-(1-p)s]} \\ &= \frac{p^{n+1}s}{[1-(1-p)s]-(p-p^{n+1})s} = \frac{p^{n+1}s}{1-(1-p^{n+1})s}. \end{aligned}$$

And the proof is complete.

Remark: In what follows you may extend the domain of $g_n(s)$ without further proof to $s \in [0, 1/(1-p^n))$. So,

$$g_n(s) = \mathbb{E}[s^{Z_n}] = \frac{p^n s}{1-(1-p^n)s} \qquad \text{for } s \in [0, 1/(1-p^n))$$

and that

$$\psi_n(t) = \mathbb{E}[e^{tZ_n}] = g_n(e^t) \qquad \text{for } t < -\log(1-p^n) = |\log(1-p^n)|.$$

c) Show that $p^n Z_n$ converges in distribution to an exponentially distributed random variable with expectation 1. (6p)
Solution: We use that the moment generating function of $p^n Z_n$ is equal to $\psi_n(p^n t) = g_n(e^{p^n t})$, for $t < p^{-n}|\log(1-p^n)]|$.
There are many ways to continue and here is one:
We define

$$h(t) = e^t - (1+t).$$

Then, for $t < p^{-n}|\log(1-p^n)]|$ we have

$$\psi_n(p^n t) = g_n(p^n e^t) = \frac{p^n e^{tp^n}}{1-(1-p^n)e^{tp^n}} = \frac{p^n[1+tp^n+h(tp^n)]}{1-(1-p^n)[1+tp^n+h(tp^n)]}$$
$$= \frac{p^n[1+tp^n+h(tp^n)]}{(1-t)p^n+tp^{2n}+(1-p^n)h(tp^n)} = \frac{1+tp^n+h(tp^n)}{(1-t)+tp^n-(1-p^n)h(tp^n)/p^n}.$$

We note that as $n \to \infty$, we have $p^n \to 0$ and $tp^n \to 0$. Because $h(x)/x \to 0$ as $x \to 0$ we have also $h(tp^n)/p^n = th(tp^n)/[tp^n] \to 0$ as $n \to \infty$. Furthermore, $p^{-n}|\log(1-p^n)]| \to 1$ as $n \to \infty$, because $\log(1-x)/x \to 1$ as $x \to 0$. This leads to $\psi_n(p^n t) \to 1/(1-t)$, for $t \in (-\infty, 1)$ (which contains an open interval around 0). The function $1/(1-t)$ is the moment generating function for an exponentially distributed random variable with expectation 1, so $p^n Z_n$ converges in distribution to an exponential distributed random variable with expectation 1.

Problem 3
Let $A_1, A_2, \cdots$ be independent events on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define

$$A := \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m.$$

That is, the A_i's happen infinitely often.
a) Prove (a special case of) the first Borel-Cantelli Lemma:

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$$

implies $\mathbb{P}(A) = 0$. (3p)
Solution: For every $i \in \mathbb{N}$, we have $A \subset \cup_{m=i}^{\infty} A_m$ and therefore we have for every $i \in \mathbb{N}$,

$$\mathbb{P}(A) = \mathbb{P}(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m) \leq \mathbb{P}(\cup_{m=i}^{\infty} A_m) \leq \sum_{m=i}^{\infty} \mathbb{P}(A_m),$$

which is decreasing in i and converges to 0 because $\sum_{i=0}^{\infty} \mathbb{P}(A_i) < \infty$.

b) Prove the second Borel-Cantelli Lemma:

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \infty$$

implies $\mathbb{P}(A) = 1$. (3p)
Solution: Note that $\sum_{n=0}^{\infty} \mathbb{P}(A_n) = \infty$ implies $\sum_{n=m}^{\infty} \mathbb{P}(A_n) = \infty$ for all $m \geq 0$. Also note that $A^c = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m^c$.
By $e^{-x} \geq 1 - x$ for $x \geq 0$ and independence, we obtain

$$\mathbb{P}(\cap_{m=n}^{\infty} A_m^c) = \prod_{m=n}^{\infty} [1 - \mathbb{P}(A_m)] \leq \prod_{m=n}^{\infty} \exp[-\mathbb{P}(A_m)] = \exp[-\sum_{m=n}^{\infty} \mathbb{P}(A_m)] = 0$$

by $\sum_{n=0}^{\infty} \mathbb{P}(A_n) = \infty$. So,

$$\mathbb{P}(A^c) = \mathbb{P}(\cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m^c) \leq \sum_{n=1}^{\infty} \mathbb{P}(\cap_{m=n}^{\infty} A_m^c) = 0.$$

c) Let $X_1, X_2, \cdots$ be independent random variables. Show that $X_n \overset{a.s.}{\to} 0$ if and only if $\sum_{i=1}^{\infty} \mathbb{P}(|X_i| > 1/k) < \infty$ for all $k \in \mathbb{N}$. (6p)
Hint: You may use without proof that

$$\{X_n \not\to 0\} = \cup_{k=1}^{\infty} \cap_{N=1}^{\infty} \cup_{n=N}^{\infty} \{(|X_n| > 1/k\}.$$

Solution: Define $A_n(1/k) := \{|X_n| > 1/k\}$. By the hint

$$\mathbb{P}(X_n \not\to 0) = \mathbb{P}(\cup_{k=1}^{\infty} \cap_{N=1}^{\infty} \cup_{n=N}^{\infty} A_n(1/k)) \leq \sum_{k=1}^{\infty} \mathbb{P}(\cap_{N=1}^{\infty} \cup_{n=N}^{\infty} A_n(1/k)).$$

From part a) we know that if $\sum_{i=1}^{\infty} \mathbb{P}(|X_i| > 1/k) < \infty$ for all $k \in \mathbb{N}$, then $\mathbb{P}(\cap_{N=1}^{\infty} \cup_{n=N}^{\infty} A_n(1/k)) = 0$ for all $k \in \mathbb{N}$. Therefore,

$$\mathbb{P}(X_n \not\to 0) \leq \sum_{k=1}^{\infty} 0 = 0.$$

If on the other hand

$$\sum_{i=1}^{\infty} \mathbb{P}(|X_i| > 1/k_0) = \infty,$$

where $k_0 \in \mathbb{N}$ we have by part b) that

$$\mathbb{P}(\text{for all } n \in \mathbb{N} \text{ there exists } m \geq n \text{ such that } |X_m| > 1/k_0) = 1.$$

That is, $\mathbb{P}(X_n \not\to 0) = 1$.

Problem 4 Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with $\mathbb{E}[X_1] = 1$, $\mathbb{P}(X_1 = 1) < 1$ and $\mathbb{P}(X_1 > \epsilon) = 1$ for some $\epsilon > 0$. For $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the σ-field generated by $X_1, X_2, \cdots, X_n$ and $\underline{\mathcal{F}}$ the corresponding filtration.

a) For $n \in \mathbb{N}$ define $Y_n = \prod_{k=1}^n X_k$. Show that $Y_1, Y_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$. (4p)

Solution: Y_n is by definition measurable with respect to $\mathcal{F}_n$.

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[Y_n X_{n+1}|\mathcal{F}_n] = Y_n \mathbb{E}[X_{n+1}|\mathcal{F}_n] = Y_n.$$

It follows also that

$$\mathbb{E}[Y_{n+1}] = \mathbb{E}[\mathbb{E}[Y_{n+1}|\mathcal{F}_n]] = \mathbb{E}[Y_n] = \cdots = \mathbb{E}[Y_1] = 1 < \infty.$$

b) Show that $\frac{1}{n}\sum_{i=1}^n \log(X_i)$ converges almost surely to a non-positive constant. (4p)

Solution: The logarithm is a concave function. So, By Jensen's inequality

$$\mathbb{E}[\log(X_1)] \leq \log(\mathbb{E}[X_1]) = \log(1) = 0.$$

Also note that $(\log x)^2 < x$ for all $x > x_0$ for some $x_0 > 0$ ($x_0 = 1$ does the job). So,

$$\mathbb{E}[(\log(X_1))^2] = \mathbb{E}[(\log(X_1))^2 \mathbb{1}(X_1 \leq x_0)] + \mathbb{E}[(\log(X_1))^2 \mathbb{1}(X_1 > x_0)]$$
$$\leq \mathbb{E}[(\log(X_1))^2 \mathbb{1}(X_1 \leq x_0)] + \mathbb{E}[X_1 \mathbb{1}(X_1 > x_0)] \leq \max_{\epsilon \leq x \leq x_0} (\log x)^2 + \mathbb{E}[X_1] < \infty.$$

So we can use the strong law of large numbers (THM 22 of cheat sheet) to show that

$$\frac{1}{n}\sum_{i=1}^n \log(X_i) \overset{a.s.}{\to} \mathbb{E}[\log[X_i] \leq 0.$$

Remark: It can be shown (and used without proof, if needed) that $\frac{1}{n}\sum_{i=1}^{n}\log(X_i)$ *converges to a strictly negative constant.*
c) Show that Y_n converges almost surely to 0.
Solution:

$$Y_n = \prod_{i=1}^{n} X_i = \prod_{i=1}^{n} e^{\log(X_i)} = e^{\sum_{i=1}^{n}\log(X_i)}.$$

So, if $\sum_{i=1}^{n}\log(X_i) \overset{a.s.}{\to} -\infty$ we are done. By $\frac{1}{n}\sum_{i=1}^{n}\log(X_i) \overset{a.s.}{\to} \mathbb{E}[\log[X_i] < 0$, this indeed holds. (4p)

Problem 5

Let $X_1, X_2, \cdots$ be independent and identically distributed random variables with

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2.$$

Let $\underline{\mathcal{F}}$ be the filtration generated by those random variables. Define $S_0 = 0$ and $S_n = \sum_{k=1}^{n} S_k$ for $n \in \mathbb{N}$. Let $a, b \in \mathbb{N}$. Let

$$T_{-a} = \inf\{k \in \mathbb{N}; S_k = -a\} \qquad \text{and} \qquad T_b = \inf\{k \in \mathbb{N}; S_k = b\},$$

be the hitting times of respectively $-a$ and b. Define $T = \min(T_{-a}, T_b)$.

a) Compute $p = \mathbb{P}(T_{-a} = T)$. (4p)

Solution: S_n is $\mathcal{F}_n$ measurable by definition. $\mathbb{E}[|S_n|] \le n < \infty$ for all n and $\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_n + X_{n+1}|\mathcal{F}_n] = S_n$. So, S_n is a martingale.
Furthermore let

$$K = \min\{k \in \mathbb{N}; T_{(a+b)(k-1)+1} = T_{(a+b)(k-1)+2} = \cdots = T_{(a+b)k} = 1\}$$

be the smallest positive integer for which $T_{(a+b)(k-1)+1} = T_{(a+b)(k-1)+2} = \cdots = T_{(a+b)k} = 1$. Because $a+b$ subsequent $+1$'s definitely brings you outside the strip (a, b) (if you were not already outside it), $T \le K(a+b)$. It is trivial to see that K is geometrically distributed with parameter $2^{-(a+b)} \in (0, 1)$. Therefore K is finite with probability 1 and has finite expectation and as consequence T is finite with probability 1 and has finite expectation. We can use Theorem 25 and 26 of the cheat sheet (noting that S_n is bounded for $n \le T$), to obtain that

$$0 = \mathbb{E}[S_1] = \mathbb{E}[S_T] = -a\mathbb{P}(T_{-a} = T) + b(1 - \mathbb{P}(T_{-a} = T)).$$

Therefore, $p = \mathbb{P}(T_{-a} = T) = b/(a+b)$.

b) Show that $Y_n = (S_n)^2 - n$ is an $\underline{\mathcal{F}}$-martingale and show that $\mathbb{E}[T] = ab$. (4p)

Solution: Y_n is $\mathcal{F}_n$ measurable, because S_n is. $\mathbb{E}[|Y_n|] \le n^2 + n < \infty$ and with

$$\begin{aligned}\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[(S_n+X_{n+1})^2-(n+1)|\mathcal{F}_n] &= (S_n^2)-(n+1)+2S_n\mathbb{E}[X_{n+1}|\mathcal{F}_n]+\mathbb{E}[(X_{n+1})^2|\mathcal{F}_n] \\ &= (S_n)^2 - (n+1) + 2S_n \times 0 + 1 = (S_n)^2 - n = Y_n.\end{aligned}$$

Using Theorem 25 of the cheat sheet we obtain

$$0 = \mathbb{E}[Y_1] = \mathbb{E}[Y_T] = pa^2 + (1-p)b^2 - \mathbb{E}[T] = ab - \mathbb{E}[T]$$

and the statement of the question follows.

c) Compute $\mathbb{E}[TS_T]$. (4p)

Hint: Find a suitable martingale. You might consider $\mathbb{E}[(S_{n+1})^3|\mathcal{F}_n]$, *to get inspiration on which martingale would be suitable.*

Solution: Follow the hint:

$$\mathbb{E}[(S_{n+1})^3|\mathcal{F}_n] = \mathbb{E}[(S_n + X_{n+1})^3|\mathcal{F}_n]$$
$$= (S_n)^3 + 3(S_n)^2\mathbb{E}[X_{n+1}] + 3S_n\mathbb{E}[(X_{n+1})^2] + \mathbb{E}[(X_{n+1})^3] = (S_n)^3 + 3S_n.$$

It follows that $Z_n = (S_n)^3 - 3nS_n$ satisfies the martingale property. Also $\mathbb{E}[|Z_n|] \leq n^3 + 3n^2$. The conditions of Theorem 25 are easily checked and we otbain

$$0 = \mathbb{E}[Z_1] = \mathbb{E}[Z_T] = -pa^3 + (1-p)b^3 - 3\mathbb{E}[TS_T]$$
$$= ba\frac{b^2 - a^2}{b + a} - 3\mathbb{E}[TS_T] = ab(b-a) - 3\mathbb{E}[TS_T].$$

So, $\mathbb{E}[TS_T] = ab(b-a)/3$.