STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT7001
TENTAMEN
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Exam Probability III

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Permissible tools: pen, paper and attached “cheat-sheet”

5 problems. Maximum of 60 points

	A	B	C	D	E
Needed points	50	45	40	35	30

Partial answers might be worth points, even if you cannot finish an answer! You are allowed to use results from the “cheat sheet” without proof, unless the proof is explicitly asked for in the question. You may also use other results discussed in the lectures or in the course material, such as the Borel-Cantelli Lemma's. If you use such a result refer to it by stating the theorem you are using or by referring to its proper name (e.g. Fatou's lemma), and explicitly check whether the conditions of the theorem are satisfied.

Throughout the exam $\mathbb{N}$ is the set of strictly positive integers, $\mathbb{Z}$ the set of all integers, $\mathbb{R}$ and all limits are for $n \to \infty$.

Problem 1

(**a**) Provide the definition of a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (3p)

(**b**) Let $\mathbb{D}$ be the set of all real numbers with finite decimal expansion. So,

$$\mathbb{D} = \{x \in \mathbb{R}; x = \pm \sum_{j=-a}^{a} d_j 10^j \text{ for some } a \in \mathbb{N} \text{ and } d_{-a}, \cdots, d_a \in \{0, 1, \cdots, 9\}\},$$

where the first and last digits of x may be zero.
(So $2.5 = 5 \times 10^{-1} + 2 \times 10^0 + 0 \times 10^1$ and $12 = 0 \times 10^{-1} + 2 \times 10^0 + 1 \times 10^1$).

Show that a random variable X is $\mathcal{F}$-measurable if and only if

$$\{\omega \in \Omega; X(\omega) \leq x\} \in \mathcal{F} \quad \text{for all } x \in \mathbb{D}.$$

(5p)

Hint: You may use Proposition 3 of cheat-sheet without further proof.

(**c**) Show that if X and Y are $\mathcal{F}$-measurable random variables, then so is $X + Y$. (4p)

Problem 2

a) A random variable X is symmetric if X has the same distribution as $-X$. Prove that if a random variable is symmetric, then the imaginary part of the characteristic function is 0. That is, show that a symmetric random variable has a real characteristic function. (4p)

Let $p \in (0,1)$ and X be a random variable satisfying

$$\mathbb{P}(X=0)=p \qquad \text{and} \qquad \mathbb{P}(X=k)=\frac{1}{2}p(1-p)^{|k|} \quad \text{for } k \in \mathbb{Z}\setminus\{0\}.$$

Note that $|X|$ has a (shifted) geometric distribution.

b) Provide the characteristic function $\varphi(t)$ of X. Here you may assume without proof that $\varphi(t)=\psi(it)$, where $i=\sqrt{-1}$ and $\psi(t)$ is the moment generating function of X. (4p)

c) For $n \in \mathbb{N} \cap (\lambda^{-1}, \infty)$, define X_n by

$$\mathbb{P}(X_n=0)=\frac{\lambda}{n} \qquad \text{and} \qquad \mathbb{P}(X_n=k)=\frac{\lambda}{2n}\left(1-\frac{\lambda}{n}\right)^{|k|} \quad \text{for } k \in \mathbb{Z}\setminus\{0\}.$$

That is, X_n is distributed as X with $p=\lambda/n$.

The sequence of random variables X_n/n converges in distribution as $n \to \infty$ (you may just accept that without proving it). Provide the distribution of that limiting random variable. (4p)

Hint: Providing the characteristic function of the limiting random variable is already worth some (but not all) points. It might be enlightning (but it is not required) to study the convergence in distribution of $|X_n|/n$.

Problem 3

Assume in all subproblems that $Z, Y, X, X_1, X_2, \cdots$ are non-negative random variables defined on the same probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$.

a) Show that the following statements are equivalent:

- $\mathbb{E}(Y) < \infty$,
- for all $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbb{E}(Y\mathbb{1}(A)) < \epsilon$ for all events $A \in \mathcal{F}$ satisfying $\mathbb{P}(A) < \delta$. (6p)

b) Let $X_n \to X$ in probability as $n \to \infty$ and let Z be a non negative random variable with $\mathbb{E}[Z] < \infty$, such that $|X_n| \leq Z$ for all n. Show that $X_n \to X$ in mean as $n \to \infty$. (6p)

Hint: First show that $\mathbb{E}[|X_n - X|] < \infty$ and then use part *a*) with $Y = |X_n - X|$ and a conveniently chosen event A.

Problem 4 Initially, a bag contains one red ball and one blue ball. At each time unit a ball is drawn uniformly at random from the bag, its colour is noted, and then returned to the bag together with a new ball of the same colour as the drawn ball. Let R_n be the number of red balls just after the n-th time unit (that is after n draws and replacements).

a) Show that $M_n := R_n/(n+2)$ constitutes a martingale which converges almost surely to some random variable M. (4p)

b) Show that R_n is uniformly distributed on $\{1, 2, \cdots, n+1\}$. (4p)

Hint: One possible approach is to use induction and show that if R_n is a random variable which is uniform on $\{1, 2, \cdots, n+1\}$, then R_{n+1} is uniform on $\{1, 2, \cdots, n+2\}$.

c) Show that M is uniformly distributed on $(0, 1)$. (4p)

Problem 5

Let $X_1, X_2, \cdots$ be independent and identically distributed random variables, with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$ and let

$$S_n = \sum_{k=1}^{n} X_k \qquad \text{and} \qquad Y_n = (S_n)^2 - n.$$

Let $\underline{\mathcal{F}}$ be the filtration generated by $X_1, X_2, \cdots$. Let a be a strictly positive integer and let $T = \min\{n \geq 1 : |S_n| = a\}$.

a) Show that $Y_1, Y_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$. (3p)

b) Show that T is a stopping time with finite mean and variance. (3p)

c) Show that $\mathbb{E}(T) = a^2$. (3p)

d) Find real constants b and c such that

$$Z_n = (S_n)^4 - 6n(S_n)^2 + bn^2 + cn$$

constitutes a martingale with respect to $\underline{\mathcal{F}}$. (3p)

Remark: This result can be used to compute $\mathbb{E}(T^2)$. You do not have to do that.

Good Luck!

Reminder

σ-algebras, probability measures and expectation

Definition 1 *The Borel σ-algebra on $\mathbb{R}$, is the smallest σ-algebra generated by the open subsets of $\mathbb{R}$. This definition can be extended to $\mathbb{R}^d$ for $d \geq 1$.*

Definition 2

$$\limsup_{n\to\infty} A_n := \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m$$

$$\liminf_{n\to\infty} A_n := \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m$$

Proposition 3 *A random variable X is $\mathcal{F}$-measurable if and only if $\{X \leq x\} := \{\omega \in \Omega : X(\omega) \leq x\}$ belongs to $\mathcal{F}$ for all $x \in \mathbb{R}$.*

Definition 4 *The distribution measure μ_X of the random variable X is the probability measure on $(\mathbb{R}, \mathcal{B})$ defined by $\mu_X(B) = \mathbb{P}(X \in B)$ for Borel sets $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra.*

Proposition 5 *If the σ-algebra $\mathcal{A}$ is generated by a finite partition $\mathcal{P}$. Then the function Y is $\mathcal{A}$ measurable if and only if Y is constant on each element of $\mathcal{P}$.*

Lemma 6 *If X, Y satisfy $\min(\mathbb{E}(X^+), \mathbb{E}(X^-)) < \infty$, then*
(i) $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$ *(linearity)*
(ii) $\mathbb{E}(X) \leq \mathbb{E}(Y)$ *if $X \leq Y$ a.s.* *(monotonicity)*

Definition 7 *Let $\mathcal{P} = \{A_1, \cdots, A_n\}$ be a finite partition, which generates the σ-algebra $\mathcal{A} \subset \mathcal{F}$, then $\mathbb{E}(X|\mathcal{A})(\omega) = \sum_{i=1}^{n} \mathbb{E}(X|A_i)\mathbb{1}(\omega \in A_i)$ for $\omega \in \Omega$*

Lemma 8 (Jensen's inequality) *We have $\mathbb{E}(\phi(X)) \geq \phi(\mathbb{E}(X))$ for convex functions ϕ.*

Characteristic functions

Definition 9 *the Characteristic function of a random variable X is the function $\varphi : \mathbb{R} \to \mathbb{C}$, defined by $\varphi_X(t) = \mathbb{E}(e^{itX}) = \mathbb{E}(\cos[tX]) + i\mathbb{E}(\sin[tx])$ where $i = \sqrt{-1}$.*

Properties of φ_X:

- $\varphi_X(0) = 1$
- $|\varphi_X(t)| \leq 1$
- $\varphi_X(-t) = \overline{\varphi_X(t)}$
- If $a, b \in \mathbb{R}$ and $Y = aX + b$ then $\varphi_Y(t) = e^{itb}\varphi_X(at)$
- If the random variables X and Y are independent, then $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$
- φ_X is real if and only if X and $-X$ have the same distribution, (X is symmetric)

Theorem 10 *Let X be a random variable with distribution function F and characteristic function φ. If F is continuous in both a and b, then*

$$F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itb} - e^{-ita}}{-it} \varphi(t)dt$$

special cases:

- If $\int_{\mathbb{R}} |\varphi(t)|dt < \infty$, then X has a continuous distribution with density
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t)dt$$
- If the distribution of X is discrete, then
$$\mathbb{P}(X = x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx} \varphi(t)dt$$

Theorem 11 *Let $\varphi^{(k)}(\cdot)$ be the k-th complex derivative of φ.*

- *If $\varphi_X^{(k)}(0)$ exists then $\mathbb{E}(|X^k|) < \infty$ if k is even and $\mathbb{E}(|X^{k-1}|) < \infty$ if k is odd*
- *if $\mathbb{E}(|X^k|) < \infty$ then $\varphi_X(t) = \sum_{j=0}^{k} \frac{\mathbb{E}(X^j)}{j!}(it)^j + o(t^k)$, where $f(x) = o(x)$ if $f(x)/x \to 0$ for $x \to 0$*

Some useful results for convergence results

Chebychev's inequality: $\mathbb{P}(|X| > x) \leq \frac{\mathbb{E}(X^2)}{x^2}$

Markov inequality: $\mathbb{P}(|X| > x) \leq \frac{\mathbb{E}(|X|^r)}{x^r}$

Hölder's inequality: For $p, q > 1$ such that $1/p + 1/q = 1$ we have

$$\mathbb{E}(|XY|) \leq [\mathbb{E}(|X|^p)]^{1/p}[\mathbb{E}(|X|^q)]^{1/q}$$

Minkovski's inequality: For $r \geq 1$ we have

$$[\mathbb{E}(|X+Y|^r)]^{1/r} \leq [\mathbb{E}(|X|^r)]^{1/r} + [\mathbb{E}(|X|^r)]^{1/r}$$

Lemma 12 (Fatou's Lemma) *Let $X_1, X_2, \cdots$ be non-negative random variables, then $\mathbb{E}(\liminf X_n) \leq \liminf \mathbb{E}(X_n)$.*

Definition 13 (Tail events) *If $X_1, X_2, \cdots$ are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{H}_n = \sigma(X_{n+1}, X_{n+2}, \cdots)$ is the smallest σ-algebra in which all random variables $X_{n+1}, X_{n+2}, \cdots$ are measurable, then $\mathcal{H}_\infty := \cap_n \mathcal{H}_n$ is called the tail σ-algebra, and events contained in it are tail events.*

Theorem 14 (Kolmogorov's zero-one law) *If $X_1, X_2, \cdots$ are independent, then all tail events $H \subset \mathcal{H}_\infty$ satisfy either $\mathbb{P}(H) = 1$ or $\mathbb{P}(H) = 0$*

Definition 15 (Uniform integrability) *A sequence of r.v. $X_1, X_2, \cdots$ is uniformly integrable if*

$$\sup_{n \geq 1} \mathbb{E}(|X_n| \mathbb{1}(|X_n| > a)) \to 0 \qquad \text{as } a \to \infty$$

Theorem 16 *Let X and $X_1, X_2, \cdots$ be random variables such that $X_n \xrightarrow{\mathbb{P}} X$ then the following statements are equivalent*

1. *$X_1, X_2, \cdots$ is uniformly integrable*
2. *$\mathbb{E}(|X_n|) < \infty$ for all n, $\mathbb{E}(|X|) < \infty$ and $X_n \xrightarrow{1} X$*
3. *$\mathbb{E}(|X_n|) < \infty$ and $\mathbb{E}(|X_n|) \to \mathbb{E}(|X|) < \infty$*

Martingales

Some Properties of martingales: Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}} = (\mathcal{F}_0, \mathcal{F}_1, \cdots)$.

- $\mathbb{E}(S_{n+m}|\mathcal{F}_n) = S_n$
- $\mathbb{E}(S_n) = \mathbb{E}(S_1)$
- $\mathbb{E}((S_n)^2)$ is non decreasing

Theorem 17 (Doob decomposition) *A $\underline{\mathcal{F}}$-submartingale $Y_0, Y_1, \cdots$ with finite means may be expressed in the form $Y_n = M_n + S_n$, where $M_1, M_2, \cdots$ is a $\underline{\mathcal{F}}$-martingale and S_n is $\mathcal{F}_{n-1}$ measurable for all n. This decomposition is unique.*

Lemma 18 (Doob-Kolmogorov inequality) *If $S_1, S_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$, then for all $\epsilon > 0$ we have $\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| \geq \epsilon\right) \leq \epsilon^{-2}\mathbb{E}((S_n)^2)$.*

Theorem 19 (Martingale convergence theorem) *If $S_1, S_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$ and $\mathbb{E}((S_n)^2) \nearrow M < \infty$, then there exists a random variable S such that $S_n \overset{a.s.}{\to} S$.*

Definition 20 (Cauchy sequence) *A sequence of real numbers $x_1, x_2, \cdots$ is a Cauchy sequence if for all $\epsilon > 0$ there exists an N such that for all $n \geq m \geq N$, we have $|x_n - x_m| < \epsilon$.*
We know that a sequence is convergent if and only if it is a Cauchy sequence.

Theorem 21 *Let $S_0, S_1, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$ such that $S_0 = 0$ and $\mathbb{E}((S_n)^2) < \infty$ for all n. Define*

$$\langle S\rangle_n = \sum_{k=1}^{n} \mathbb{E}((S_k - S_{k-1})^2|\mathcal{F}_{k-1}) \qquad \textit{and} \qquad \langle S\rangle_\infty = \lim_{n\to\infty} \langle S\rangle_n.$$

Let $f \geq 1$ be a given increasing function satisfying $\int_0^\infty [f(x)]^{-2}dx < \infty$. Then,
(i) On $\{\omega : \langle S(\omega)\rangle_\infty < \infty\}$ $S_n \overset{a.s.}{\to} S$ for some random variable S
(ii) On $\{\omega : \langle S(\omega)\rangle_\infty = \infty\}$, $S_n/f(\langle S\rangle_n) \overset{a.s.}{\to} 0$

Theorem 22 (Strong Law of Large Numbers) *Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$ and define $S_0 = 0$ and $S_n = \sum_{k=1}^{n}(X_k - \mu)$ for $n \geq 1$. Then $\frac{S_n}{n} \overset{a.s.}{\to} 0$.*

Theorem 23 (Martingale Central Limit theorem) *$S_0, S_1, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$, with $S_0 = 0$ and $\mathbb{E}((S_n)^2) < \infty$ for all n. Assume that $n^{-1}\langle S\rangle_n \xrightarrow{\mathbb{P}} \sigma^2 > 0$ and for all $\epsilon > 0$*

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}((S_k - S_{k-1})^2 \mathbb{1}((S_k - S_{k-1})^2 > \epsilon n)) \to 0.$$

Then, $\frac{1}{\sqrt{n\sigma^2}} S_n \xrightarrow{d} \mathcal{N}(0, 1)$

Theorem 24 (Optional stopping I) *Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$. If T is an a.s. bounded stopping time for $\underline{\mathcal{F}}$ (i.e. $\mathbb{P}(T \leq a) = 1$ for some $a \geq 0$), then $\mathbb{E}(S_T) = \mathbb{E}(S_1)$.*

Theorem 25 (Optional stopping II) *Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$ and T a stopping time for $\underline{\mathcal{F}}$. Then $\mathbb{E}(S_T) = \mathbb{E}(S_1)$, if the following conditions hold*

- $\mathbb{P}(T < \infty) = 1$,
- $\mathbb{E}(|S_T|) < \infty$,
- $\mathbb{E}(S_n \mathbb{1}(T > n)) \to 0$ *as* $n \to \infty$.

Theorem 26 (Optional Stopping III) *Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$ and T a stopping time for $\underline{\mathcal{F}}$. Then $\mathbb{E}(S_T) = \mathbb{E}(S_1)$, if the following conditions hold*

- $\mathbb{E}(T) < \infty$,
- $\mathbb{E}(|S_{n+1} - S_n||\mathcal{F}_n) \leq K$ *for all* $n < T$ *and some* $K > 0$

Wald's equation and identity: If $X_1, X_2, \cdots$ are i.i.d. random variables with $\mathbb{E}(X_1) = \mu < \infty$ and $S_n = \sum_{k=1}^{n} X_k$ and T is a stopping time satisfying $\mathbb{E}(T) < \infty$, then $\mathbb{E}(S_T) = \mu\mathbb{E}(T)$.
If in addition there exists a $h > 0$ such that $M(t) = \mathbb{E}(e^{tX_1}) < \infty$ for all $|t| < h$ and $M(t) > 1$ and $|S_n| < C$ for some constant $C > 0$ and all $n \leq T$, then $\mathbb{E}(e^{tS_T}[M(t)]^{-T}) = 1$.