Solutions exam Probability III

October 27, 2020

Problem 1
(**a**) Provide the definition of a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (3p)

Solution: A random variable is a function $X : \Omega \to \mathbb{R}$ with the property that the set $\{\omega \in \Omega : X(\omega) \in B\}$ belongs to $\mathcal{F}$ for each Borel set $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$.

(**b**) Let $\mathbb{D}$ be the set of all real numbers with finite decimal expansion. So,

$$\mathbb{D} = \{x \in \mathbb{R}; x = \pm \sum_{j=-a}^{a} d_j 10^j \text{ for some } a \in \mathbb{N} \text{ and } d_{-a}, \cdots, d_a \in \{0, 1, \cdots, 9\}\},$$

where the first and last digits of x may be zero.
(So $2.5 = 5 \times 10^{-1} + 2 \times 10^0 + 0 \times 10^1$ and $12 = 0 \times 10^{-1} + 2 \times 10^0 + 1 \times 10^1$).
Show that a random variable X is $\mathcal{F}$-measurable if and only if $\{\omega \in \Omega; X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{D}$. (5p)

Solution: By Proposition 3 of the cheat sheet we know that a random variable X is $\mathcal{F}$-measurable if and only if $\{\omega \in \Omega; X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$. It follows immediately from $\mathbb{D} \subset \mathbb{R}$ that $\{\omega \in \Omega; X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{D}$ if $\{\omega \in \Omega; X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$.
To prove the other implication, define $b(x) = \min\{j \in \mathbb{N}; 10^j > x\}$ for $x \in \mathbb{R}$. Further define

$$\mathbb{D}_a = \{x \in \mathbb{D}; x = \pm \sum_{j=-a}^{a} d_j 10^j \text{ for some } d_{-a}, \cdots, d_a \in \{0, 1, \cdots, 9\}\},$$

For $a \geq b(x)$ define $x_a = \min\{y \in \mathbb{D}_a; y \geq x\}$. Then we note that x_a is a decreasing sequence converging to x (note that $|x_a - x| < 10^{-a} \to 0$ as $a \to \infty$. So, for all $x \in \mathbb{R}$ we obtain that

$$\{\omega \in \Omega; X(\omega) \leq x\} = \cap_{a=b(x)}^{\infty} \{\omega \in \Omega; X(\omega) \leq \{\omega \in \Omega; X(\omega) \leq x_a\}\}.$$

If we assume that all elements of the intersection are in $\mathcal{F}$, then so is $\{\omega \in \Omega; X(\omega) \leq x\}$, since a σ-algebra is closed under countable intersections and we are ready.

(**c**) Show that if X and Y are $\mathcal{F}$-measurable random variables, then so is $X + Y$. (4p)

Solution: By part b we have to show that $\{\omega \in \Omega; X(\omega) + Y(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{D}$. Note that for all $y \in \mathbb{D}$ we have

$$\{\omega \in \Omega; X(\omega) = y\} = \{\omega \in \Omega; X(\omega) \leq y\} \cap (\cap_{k=1}^{\infty} \{\omega \in \Omega; X(\omega) \leq y - 10^{-k}\}^C,$$

which is in $\mathcal{F}$, because complements and countable intersections of elements of a σ-algebra are in the σ-algebra.
Now note that for all $x \in \mathbb{D}$

$$\{\omega \in \Omega; X(\omega) + Y(\omega) \leq x\} = \cap_{y \in \mathbb{D}} \left(\{\omega \in \Omega; X(\omega) = y\} \cap \{\omega \in \Omega; Y(\omega) \leq x - y\}\right).$$

By construction $\mathbb{D}$ is a countable set and on the right hand side all separate sets in the intersection are elements of $\mathcal{F}$, since y and x (and therefore $x - y$) are elements of $\mathbb{D}$. So the left hand side is also in $\mathcal{F}$.

Problem 2

a) A random variable X is symmetric if X has the same distribution as $-X$. Prove that if a random variable is symmetric, then the imaginary part of the characteristic function is 0. That is, show that a symmetric random variable has a real characteristic function. (4p)

Solution: The characteristic function of X is given by

$$\varphi(t) = \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)]$$

and the characteristic function of $-X$ is given by

$$\mathbb{E}[\cos(t[-X])] + i\mathbb{E}[\sin(t[-X])] = \mathbb{E}[\cos(tX)] - i\mathbb{E}[\sin(tX)],$$

where we used the symmetry of the cosine function and the anti-symmetry of the sine function. If X has the same distribution as $-X$ then they have the same characteristic function and therefore $i\mathbb{E}[\sin(tX)] = -i\mathbb{E}[\sin(tX)]$ for all t, which implies $\mathbb{E}[\sin(tX)] = 0$ and the result follows.

Let $p \in (0,1)$ and X be a random variable satisfying

$$\mathbb{P}(X = 0) = p \qquad \text{and} \qquad \mathbb{P}(X = k) = \frac{1}{2}p(1-p)^{|k|} \quad \text{for } k \in \mathbb{Z} \setminus \{0\}.$$

Note that $|X|$ has a (shifted) geometric distribution.

b) Provide the characteristic function $\varphi(t)$ of X. Here you may assume without proof that $\varphi(t) = \psi(it)$, where $i = \sqrt{-1}$ and $\psi(t)$ is the moment generating function of X. (4p)

Solution: First we compute the moment generating function

$$\begin{aligned}
\psi(t) = \mathbb{E}[e^{tX}] = pe^{t\cdot 0} + \sum_{k=1}^{\infty} \frac{1}{2}p(1-p)^k e^{tk} + \sum_{k=-\infty}^{-1} \frac{1}{2}p(1-p)^{-k} e^{tk} \\
= \sum_{k=0}^{\infty} \frac{1}{2}p(1-p)^k e^{tk} + \sum_{k=-\infty}^{0} \frac{1}{2}p(1-p)^{-k} e^{tk} \\
= \frac{p}{2}\frac{1}{1-(1-p)e^t} + \frac{p}{2}\frac{1}{1-(1-p)e^{-t}} \\
= \frac{p}{2}\frac{2-(1-p)(e^{-t}+e^t)}{1+(1-p)^2-(1-p)(e^t+e^{-t})} = p\frac{1-(1-p)\frac{e^{-t}+e^t}{2}}{1+(1-p)^2-2(1-p)\frac{e^t+e^{-t}}{2}}
\end{aligned}$$

Using the hint we obtain that

$$\varphi(t) = p\frac{1-(1-p)\frac{e^{-it}+e^{it}}{2}}{1+(1-p)^2-2(1-p)\frac{e^{it}+e^{-it}}{2}}.$$

Further noting that $\frac{e^{it}+e^{-it}}{2} = \frac{\cos t + i\sin t + \cos t - i \sin t}{2} = \cos t$, we obtain that

$$\varphi(t) = p\frac{1-(1-p)\cos t}{1+(1-p)^2 - 2(1-p)\cos t}.$$

c) For $n \in \mathbb{N} \cap (\lambda^{-1}, \infty)$, define X_n by

$$\mathbb{P}(X_n = 0) = \frac{\lambda}{n} \qquad \text{and} \qquad \mathbb{P}(X_n = k) = \frac{\lambda}{2n}\left(1 - \frac{\lambda}{n}\right)^{|k|} \quad \text{for } k \in \mathbb{Z} \setminus \{0\}.$$

That is, X_n is distributed as X with $p = \lambda/n$.
The sequence of random variables X_n/n converges in distribution as $n \to \infty$ (you may just accept that without proving it). Provide the distribution of that limiting random variable. (4p)
Hint: Providing the characteristic function of the limiting random variable is already worth some (but not all) points. It might be enlightning (but it is not required) to study the convergence in distribution of $|X_n|/n$.

Solution: Let $\varphi_n(t)$ be the characteristic function of X_n, which by part b

$$\varphi_n(t) = \frac{\lambda}{n}\frac{1-(1-\lambda/n)\cos t}{1+(1-\lambda/n)^2 - 2(1-\lambda/n)\cos t}.$$

Standard properties of characteristic functions then give that the characteristic function of X_n/n is given by $\varphi_n(t/n)$, which is

$$\varphi_n(t/n) = \frac{\lambda}{n}\frac{1-(1-\lambda/n)\cos[t/n]}{1+(1-\lambda/n)^2 - 2(1-\lambda/n)\cos[t/n]}.$$

Note that using the Taylor expansion of the cosine,

$$\cos[t/n] = 1 - \frac{1}{2}\frac{t^2}{n^2} + o(1/n^2).$$

So,

$$\varphi_n(t/n) = \frac{\lambda}{n}\frac{1-(1-\lambda/n)(1-\frac{t^2}{2n^2}) + o(1/n^2)}{1+(1-\lambda/n)^2 - 2(1-\lambda/n)(1-\frac{t^2}{2n^2}) + o(1/n^2)}$$
$$= \frac{\lambda}{n}\frac{\lambda/n + (1-\lambda/n)\frac{t^2}{2n^2} + o(1/n^2)}{(\lambda/n)^2 + 2(1-\lambda/n)\frac{t^2}{2n^2} + o(1/n^2)} = \frac{\lambda^2 + o(1/n)}{\lambda^2 + t^2 + o(1/n)} \to \frac{\lambda^2}{\lambda^2+t^2}.$$

This is the Characteristic function of the random variable with density function $\frac{\lambda}{2}e^{-\lambda|x|}$, which is the density function of a "symmetrized" exponential.

Problem 3
Assume in all subproblems that $Z, Y, X, X_1, X_2, \cdots$ are non-negative random variables defined on the same probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$.
a) Show that the following statements are equivalent:

- $\mathbb{E}(Y) < \infty$,
- for all $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbb{E}(Y\mathbb{1}(A)) < \epsilon$ for all events $A \in \mathcal{F}$ satisfying $\mathbb{P}(A) < \delta$. (6p)

Solution: Suppose that for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that $\mathbb{E}(Y\mathbb{1}_A) < \epsilon$ for all A satisfying $\mathbb{P}(A) < \delta$. Let ϵ and $x > 0$ be such that $\mathbb{P}(Y > x) < \delta(\epsilon)$. Then for all $y > x$ we have

$$\mathbb{E}(Y) = \mathbb{E}(Y\mathbb{1}(Y \leq y)) + \mathbb{E}(Y\mathbb{1}(Y > y)) \leq y\mathbb{P}(Y \leq y) + \epsilon \leq y + \epsilon < \infty.$$

If on the other hand $\mathbb{E}(Y) < \infty$, then $\mathbb{E}(Y\mathbb{1}(Y > x)) \to 0$. So, there exists y such that $\mathbb{E}(Y\mathbb{1}(Y > y) < \epsilon/2$. Now note that

$$\begin{aligned}\mathbb{E}(Y\mathbb{1}(A)) &= \mathbb{E}(Y\mathbb{1}(A \cap Y > y)) + \mathbb{E}(Y\mathbb{1}(A \cap Y \leq y)) \\ &\leq \mathbb{E}(Y\mathbb{1}(Y > y)) + y\mathbb{P}(A \cap Y \leq y) \leq \epsilon/2 + y\mathbb{P}(A).\end{aligned}$$

The theorem follows by chosing $\delta = \epsilon/(2y)$.

b) Let $X_n \to X$ in probability as $n \to \infty$ and let Z be a non negative random variable with $\mathbb{E}[Z] < \infty$, such that $|X_n| \leq Z$ for all n. Show that $X_n \to X$ in mean as $n \to \infty$. (6p)
Hint: First show that $\mathbb{E}[|X_n - X|] < \infty$ and then use part a) with $Y = |X_n - X|$ and a conveniently chosen event A.

Solution: $X_n \xrightarrow{\mathbb{P}} X$ implies $\mathbb{P}(|X_n - X| \geq \kappa) \to 0$ for all $\kappa > 0$. Therefore, $X_n \xrightarrow{\mathbb{P}} X$ and $|X_n| < Z$ implies that $\mathbb{P}(|X| \geq Z + \kappa) = 0$ for all $\kappa > 0$, and therefore $|X|$ is almost surely less than or equal to Z. This implies that we have $|X_n - X| \leq 2Z$ almost surely. Which in turn implies that $\mathbb{E}(|X_n - X|) < \infty$.
Observe further that

$$\begin{aligned}\mathbb{E}(|X_n - X|) &= \mathbb{E}(|X_n - X|\mathbb{1}(|X_n - X| > \kappa)) + \mathbb{E}(|X_n - X|\mathbb{1}(|X_n - X| \leq \kappa)) \\ &\leq \kappa + \mathbb{E}(|X_n - X|\mathbb{1}(|X_n - X| > \kappa))\end{aligned}$$

Note that $\mathbb{P}(|X_n - X| > \kappa) \to 0$, so there is an integer N such that for all $n > N$ we have $\mathbb{P}(|X_n - X| > \kappa) < \delta$.
By part (**a**) and $\mathbb{E}(|X_n - X|) < \infty$, this implies

$$\mathbb{E}(|X_n - X|) \leq \kappa + \epsilon.$$

Sending κ and ϵ to 0 gives the desired result.

Problem 4 Initially, a bag contains one red ball and one blue ball. At each time unit a ball is drawn uniformly at random from the bag, its colour is noted, and then returned to the bag together with a new ball of the same colour as the drawn ball. Let R_n be the number of red balls just after the n-th time unit (that is after n draws and replacements).

a) Show that $M_n := R_n/(n+2)$ constitutes a martingale which converges almost surely to some random variable M. (4p)

Solution: By definition we have $|M_n| = M_n \leq 1 < \infty$. Furthermore, because the probability of drawing a red ball in step n is M_n we obtain

$$\mathbb{E}[M_{n+1}|M_n] = M_n \frac{R_n+1}{n+3} + (1-M_n)\frac{R_n}{n+3} = \frac{R_n}{n+3} + \frac{M_n}{n+3} = \frac{R_n}{n+3} + \frac{R_n}{(n+2)(n+3)} = \frac{R_n}{n+2} = M_n$$

and the conditions for a martingale are checked. We then can use the martingale convergence theorem, noting that $\mathbb{E}[(M_n)^2] \leq 1 < \infty$ for all n.

b) Show that R_n is uniformly distributed on $\{1, 2, \cdots, n+1\}$. (4p)

Solution: It is clear that $\mathbb{P}(R_1 = 1) = \mathbb{P}(R_1 = 2) = 1/2$ because the probability that the first draw gives a red ball is $1/2$.
Now assume that $\mathbb{P}(R_n = k) = 1/(n+1)$ for $k \in \{1, 2, \cdots, n+1\}$. Then

$$\mathbb{P}(R_{n+1} = k) = \sum_{j=1}^{n+1} \mathbb{P}(R_{n+1} = k | R_n = j)\mathbb{P}(R_n = j)$$

by the law of total probability. Also note that $\mathbb{P}(R_{n+1} = k|R_n = j)$ is zero if $j \notin \{k-1, k\}$, while

$$\mathbb{P}(R_{n+1} = k|R_n = k-1) = \frac{k-1}{n+2} \quad \text{and} \quad \mathbb{P}(R_{n+1} = k|R_n = k) = 1 - \frac{k}{n+2}.$$

So, for $k \in \{1, 2, \cdots, n+2\}$

$$\mathbb{P}(R_{n+1} = k) = \frac{k-1}{n+2}\frac{1}{n+1} + \left(1 - \frac{k}{n+2}\right)\frac{1}{n+1} = \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} = \frac{1}{n+2},$$

as desired. Note that for $k = 1$ and $k = n+2$ one of the summands in the second term is 0.

c) Show that M is uniformly distributed on $(0, 1)$. (4p)

Solution: For $x \in (0, 1)$,

$$\mathbb{P}(M_n \leq x) = \mathbb{P}(R_n \leq (n+2)x) = \frac{1}{n+2}\sum_{k=1}^{n+1} \mathbb{1}(k \leq (n+2)x) \to x$$

by part b)

Which means that $M_n \xrightarrow{d} M$, where M is uniformly distributed on $(0, 1)$. Therefore M from part a) is uniformly distributed.

Problem 5
Let $X_1, X_2, \cdots$ be independent and identically distributed random variables, with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$ and let

$$S_n = \sum_{k=1}^{n} X_k \qquad \text{and} \qquad Y_n = (S_n)^2 - n.$$

Let $\underline{\mathcal{F}}$ be the filtration generated by $X_1, X_2, \cdots$. Let a be a strictly positive integer and let $T = \min\{n \geq 1 : |S_n| = a\}$.
a) Show that $Y_1, Y_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$. (3p)

Solution: First Y_n is defined in terms of $X_1, X_2, \cdots, X_n$ and n, therefore it is $\mathcal{F}_n$ measurable. Furthermore, $\mathbb{E}(|Y_n|) \leq n^2 + n < \infty$ for all $n \geq 1$. Finally,

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}((S_n)^2 + 2S_nX_{n+1} + (X_{n+1})^2 - (n+1)|\mathcal{F}_n)$$
$$= (S_n)^2 + 2S_n\mathbb{E}(X_{n+1}) + \mathbb{E}((X_{n+1})^2) - (n+1) = (S_n)^2 + S_n \times 0 + 1 - (n+1) = (S_n)^2 - n = Y_n$$

b) Show that T is a stopping time with finite mean and variance. (3p)

Solution: T is a stopping time, since the event $\{T = n\}$ only depends on $X_1, X_2, \cdots, X_n$. Furthermore, for $n \geq 1$, let A_n be the event

$$\{X_{(n-1)2a+1} = 1, X_{(n-1)2a+2} = 1, \cdots, X_{(n)2a} = 1\},$$

which is the event that from time $(n-1)2a + 1$ up to and including time $2an$ the random walk only increases. Note that if A_n occurs then $T \leq 2an$. In particular for $T_1 = \min\{n \geq 1 : A_n \text{ occurs}\}$, we have that $T \leq 2aT_1$. Furthermore Since T_1 is geometrically distributed with parameter 2^{-2a}, T_1 and therefore T have finite first and second moment.

c) Show that $\mathbb{E}(T) = a^2$. (3p)

Solution: We use the second optional stopping theorem.
$\mathbb{E}(|Y_T|) \leq a^2 + \mathbb{E}(T) < \infty$ and

$$|\mathbb{E}(Y_n|T > n)\mathbb{P}(T > n)| \leq |a^2\mathbb{P}(T > n)| + |\mathbb{E}(T|T > n)\mathbb{P}(T > n)| \leq a^2\mathbb{P}(T > n) + \mathbb{E}(T\mathbb{1}(T > n)) \to 0$$

by the fact that $\mathbb{E}(T) < \infty$. Therefore $0 = \mathbb{E}(Y_1) = \mathbb{E}(Y_T) = a^2 - \mathbb{E}(T)$ and $\mathbb{E}(T) = a^2$ follows.

d) Find real constants b and c such that

$$Z_n = (S_n)^4 - 6n(S_n)^2 + bn^2 + cn$$

constitutes a martingale with respect to $\underline{\mathcal{F}}$. (3p)

Remark: This result can be used to compute $\mathbb{E}(T^2)$. You do not have to do that. In Home exam the first and second summand on the Right Hand Side were multiplied by 2.

Solution: First, $\mathbb{E}(|Z_n|) \leq n^4 + 6n^3 + |b|n^2 + |c|n < \infty$. Secondly,

$$\begin{aligned}
\mathbb{E}(Z_{n+1}|\mathcal{F}_n) &= \mathbb{E}((S_n+X_{n+1})^4 - 6(n+1)(S_n+X_{n+1})^2 + b(n+1)^2 + c(n+1)|\mathcal{F}_n) \\
&= \mathbb{E}((S_n^4 + 4S_n^3X_{n+1} + 6S_n^2X_{n+1}^2 + 4S_nX_{n+1}^3 + X_{n+1}^4) \\
&- 6(n+1)(S_n^2 + 2S_nX_{n+1} + X_{n+1}^2) + b(n+1)^2 + c(n+1)|\mathcal{F}_n) \\
&= S_n^4 + 4S_n^3\mathbb{E}(X_{n+1}) + 6S_n^2\mathbb{E}(X_{n+1}^2) + 4S_n\mathbb{E}(X_{n+1}^3) + \mathbb{E}(X_{n+1}^4)) \\
&- 6(n+1)(S_n^2 + 2S_n\mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2)) + b(n+1)^2 + c(n+1) \\
&= S_n^4 + 6S_n^2 + 1 - 6(n+1)(S_n^2 + 1) + b(n+1)^2 + c(n+1) \\
&= S_n^4 - 6nS_n^2 + bn^2 + cn + (2b-6)n + (b+c-5) = Z_n + (2b-6)n + (b+c-5)
\end{aligned}$$

So Z_n constitutes a martingale if $2b - 6 = 0$ and $b + c - 5 = 0$. That is, if $b = 3$ and $c = 2$.