STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT7001
TENTAMEN
November 25, 2020

Second exam Probability III

November 25, 2020 kl. 9–15

Examiner: Pieter Trapman, tel. 07 22928174, `ptrapman@math.su.se`
Permissible tools: pen, paper and attached "cheat-sheet". Not permissable are any consultation of books, notes, internet of any communication on the exam with other people, apart from the lecturer.
Mode of submission: Send your readable solutions as a PDF document (either scanned in with a good tool or created through Latex or similar) to `ptrapman@math.su.se` by email AND submit it at the tool at the bottom of course homepage `https://kurser.math.su.se/course/view.php?id=933`

Five problems. Maximum of 60 points

	A	B	C	D	E
Needed points	50	45	40	35	30

Partial answers might be worth points, even if you cannot finish an answer! You are allowed to use results from the "cheat sheet" without proof, unless the proof is explicitly asked for in the question. You may also use other results discussed in the lectures or in the course material, such as the Borel-Cantelli Lemma's. If you use such a result refer to it by stating the theorem you are using or by referring to its proper name (e.g. Fatou's lemma), and explicitly check whether the conditions of the theorem are satisfied.

Throughout the exam $\mathbb{N}$ is the set of strictly positive integers, $\mathbb{N}_0$ is the set of non-negative integers, $\mathbb{Z}$ the set of all integers, $\mathbb{R}$ the real numbers and all limits are for $n \to \infty$, unless specified otherwise.

Prequel

State and sign that you have obtained your solutions of this exam without consulting other people, the internet, books, notes etc. during the time of the exam, other than for contacting the teacher, either for questions on clarification or for submitting the solutions.

Problem 1

(**a**) Provide the definition of a σ-algebra defined on the sample space Ω.(3p)

(**b**) Let $\mathcal{F}$ be the smallest σ-algebra on $\Omega = (0,1)$ containing all open intervals contained in $(0,1)$. Show that every single point $x \in (0,1)$ is an element of $\mathcal{F}$. (4p)

(**c**) Let $\mathcal{G}$ be the smallest σ-algebra on $\Omega = (0,1)$ containing all single points in $(0,1)$. Show that $\mathcal{G} \neq \mathcal{F}$. That is, give an element of $\mathcal{F}$ which is not in $\mathcal{G}$ and explain why this element is not in $\mathcal{G}$. (5p)

Problem 2

a) Let X be an exponential distributed random variable with expectation $1/\lambda$. That is, $\mathbb{P}(X > x) = e^{-\lambda x}$ for all $x \geq 0$. Provide the characteristic function $\varphi_X(t)$ of X. (3p)

Hint: You may assume without proof that $\varphi_X(t) = \psi(it)$, where $i = \sqrt{-1}$ and $\psi(t)$ is the moment generating function of X.

b) Let Y be a random variable with distribution defined through

$$\mathbb{P}(Y = 1) = \mathbb{P}(Y = -1) = 1/2.$$

Let $Z = YX$. Deduce what $\varphi_Z(t)$, the characteristic function of Z, is. (3p)

Remark: The characteristic function is $\varphi_Z(t) = \frac{\lambda^2}{\lambda^2+t^2}$. If you did not manage to deduce that, you can still obtain some points for part b), for correct elements in the deduction and you can still use the correct answer for part c).

Let $Y_1, Y_2, \cdots$ and $X_1, X_2, \cdots$ be independent random variables, where $Y_1, Y_2, \cdots$ are all distributed as Y and for $k \in \mathbb{N}$, X_k is exponentially distributed with expectation $1/\sqrt{k}$. For $k \in \mathbb{N}$, define $Z_k = Y_k X_k$.

c) Provide the characteristic function of $W_n = \frac{1}{\sqrt{\log n}} \sum_{k=1}^{n} Z_k$. Use this to show that W_n converges in distribution to some random variable as $n \to \infty$. What is the distribution of this random variable? (6p)

Hint: You may use without proof that for a positive function $c(n) \geq 0$, which decreases to 0 as $n \to \infty$, we have

$$n^{-c(n)} \prod_{k=1}^{n} \left(1 + \frac{c(n)}{k}\right) \to 1.$$

Problem 3

Assume in all subproblems that $X, X_1, X_2, \cdots$ are random variables defined on the same probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$. Assume in both part a) and b) that $X_n \stackrel{\mathbb{P}}{\to} X$.

a) Show that there exists a non-random strictly increasing sequence of positive integers $n_1, n_2, \cdots$, such that $X_{n_k} \stackrel{a.s.}{\to} X$ as $k \to \infty$. (6p)

Hint: You may use without proof that if $\mathbb{1}(|X_n - X| > \epsilon) \stackrel{a.s.}{\to} 0$ for every $\epsilon > 0$ then $X_n \stackrel{a.s.}{\to} X$.

You do not have to use part a) in part b) below.

b) Show that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $g(X_n) \stackrel{\mathbb{P}}{\to} g(X)$.(6p)

Hint: Note that if $g : \mathbb{R} \to \mathbb{R}$ is continuous, then it is uniform continuous on any interval $[-M, M]$ with $M \in (0, \infty)$. That is, for all $\epsilon > 0$ there exist $\delta > 0$ such that $|g(x) - g(y)| < \epsilon$ if $|x - y| < \delta$ and $x, y \in [-M, M]$.

Problem 4

Let $\lambda > 1$ be constant. For $i, j \in \mathbb{N}_0$ let X_{ij} be independent and identically distributed random variables with a Poisson distribution with expectation λ. That is,

$$\mathbb{P}(X_{ij} = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } k \in \mathbb{N}_0.$$

Define $Z_0 = 1$. For $i \in \mathbb{N}_0$ define

$$Z_{i+1} = \sum_{j=1}^{Z_i} X_{ij},$$

where we define $\sum_{j=1}^{0} = 0$.

a) Show (e.g. by induction) that

$$\mathbb{E}[Z_n] = \lambda^n \quad \text{and} \quad \mathbb{E}[(Z_n)^2] = \lambda^n \frac{\lambda^{n+1} - 1}{\lambda - 1} \quad \text{for all } n \in \mathbb{N}.$$

(4p)

b) Show that

$$W_n := \frac{1}{\lambda^n} Z_n$$

converges almost surely to some random variable W. Furthermore, show that $\mathbb{P}(W < \infty) = 1$. (4p)

c) Show that

$$V_n = \lambda^{-n} \sum_{i=1}^{n} Z_i$$

converges almost surely to $V = \left(\sum_{i=0}^{\infty} \lambda^{-i}\right) W = \frac{\lambda}{\lambda - 1} W$. (4p)

Problem 5

Let $p \in (0,1)$ be constant and $X_1, X_2, \cdots$ be independent and identically distributed random variables, satisfying

$$\mathbb{P}(X_1 = 1) = p \quad \text{and} \quad \mathbb{P}(X_1 = -1) = 1 - p.$$

For reasons of convenience set $X_0 = -1$. For $n \in \mathbb{N}$ let $K_n = 0$ if $X_n = -1$ and otherwise let K_n be the length of the sequence of consecutive $+1$'s ending at n. That is, $K_n = \min\{j \in \mathbb{N}_0; X_{n-j} = -1\}$.

Let $L \in \mathbb{N}$ be a given integer. We are interested in $T = \min\{n \in \mathbb{N}; K_n = L\}$, which is the first time a sequence of L consecutive $+1$'s appears.

To study T we can use the loss (or gain, if negative) of a casino in which the folowing happens:

1. At time 0 the loss of the casino is 0.
2. At each positive integer time point one new gambler arrives with gambling capital 1 SEK which he or she puts immediately at stake.
3. At time $k \in \mathbb{N}$, if $X_k = -1$, all gamblers which were still in the casino and arrived at time k or before, leave the casino empty handed.
4. At time $k \in \mathbb{N}$, if $X_k = 1$, all gamblers present at the casino (including the one that arrived at time k) multiply their capital instantly by a factor $1/p$, which they will put again at stake at time $k + 1$.

So, at time n (immediately after the arrival of the n-th gambler and the n-th bet) the number of gamblers in the casino is K_n.

a) Show that the loss of the casino at time n (when the new arrival and time n gambling has already occured) is given by

$$M_n = \frac{p^{-K_n} - 1}{1 - p} - n$$

and show that $M_1, M_2, \cdots$ constitutes a martingale with respect to $\underline{\mathcal{F}}$, the filtration generated by $X_1, X_2, \cdots$. (4p)

b) Compute $\mathbb{E}[T]$. You may use without proof that T is a stopping time with respect to $\underline{\mathcal{F}}$ and that $\mathbb{E}[T] < \infty$. (3p)

c) Let $s \in [0,1]$ be a constant. Compute $\mathbb{E}[s^T]$. (5p)

Hint: A way to solve this might be to adapt step 2 above in such a way that for all $n \in \mathbb{N}$, the customer arriving at time n has initial gambling capital s^{n-1}. Note that partial answers might be worth points.

Good Luck!

Reminder

σ-algebras, probability measures and expectation

Definition 1 *The Borel σ-algebra on $\mathbb{R}$, is the smallest σ-algebra generated by the open subsets of $\mathbb{R}$. This definition can be extended to $\mathbb{R}^d$ for $d \geq 1$.*

Definition 2

$$\limsup_{n\to\infty} A_n := \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m$$
$$\liminf_{n\to\infty} A_n := \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m$$

Proposition 3 *A random variable X is $\mathcal{F}$-measurable if and only if $\{X \leq x\} := \{\omega \in \Omega : X(\omega) \leq x\}$ belongs to $\mathcal{F}$ for all $x \in \mathbb{R}$.*

Definition 4 *The distribution measure μ_X of the random variable X is the probability measure on $(\mathbb{R}, \mathcal{B})$ defined by $\mu_X(B) = \mathbb{P}(X \in B)$ for Borel sets $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra.*

Proposition 5 *If the σ-algebra $\mathcal{A}$ is generated by a finite partition $\mathcal{P}$. Then the function Y is $\mathcal{A}$ measurable if and only if Y is constant on each element of $\mathcal{P}$.*

Lemma 6 *If X, Y satisfy $\min(\mathbb{E}(X^+), \mathbb{E}(X^-)) < \infty$, then*
(i) $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$ (linearity)
(ii) $\mathbb{E}(X) \leq \mathbb{E}(Y)$ if $X \leq Y$ a.s. (monotonicity)

Definition 7 *Let $\mathcal{P} = \{A_1, \cdots, A_n\}$ be a finite partition, which generates the σ-algebra $\mathcal{A} \subset \mathcal{F}$, then $\mathbb{E}(X|\mathcal{A})(\omega) = \sum_{i=1}^{n} \mathbb{E}(X|A_i)\mathbb{1}(\omega \in A_i)$ for $\omega \in \Omega$*

Lemma 8 (Jensen's inequality) *We have $\mathbb{E}(\phi(X)) \geq \phi(\mathbb{E}(X))$ for convex functions ϕ.*

Characteristic functions

Definition 9 *the Characteristic function of a random variable X is the function $\varphi : \mathbb{R} \to \mathbb{C}$, defined by $\varphi_X(t) = \mathbb{E}(e^{itX}) = \mathbb{E}(\cos[tX]) + i\mathbb{E}(\sin[tx])$ where $i = \sqrt{-1}$.*

Properties of φ_X:

- $\varphi_X(0) = 1$
- $|\varphi_X(t)| \leq 1$
- $\varphi_X(-t) = \overline{\varphi_X(t)}$
- If $a, b \in \mathbb{R}$ and $Y = aX + b$ then $\varphi_Y(t) = e^{itb}\varphi_X(at)$
- If the random variables X and Y are independent, then $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$
- φ_X is real if and only if X and $-X$ have the same distribution, (X is symmetric)

Theorem 10 *Let X be a random variable with distribution function F and characteristic function φ. If F is continuous in both a and b, then*

$$F(b) - F(a) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itb} - e^{-ita}}{-it} \varphi(t) dt$$

special cases:

- If $\int_{\mathbb{R}} |\varphi(t)| dt < \infty$, then X has a continuous distribution with density
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt$$
- If the distribution of X is discrete, then
$$\mathbb{P}(X = x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx} \varphi(t) dt$$

Theorem 11 *Let $\varphi^{(k)}(\cdot)$ be the k-th complex derivative of φ.*

- *If $\varphi_X^{(k)}(0)$ exists then $\mathbb{E}(|X^k|) < \infty$ if k is even and $\mathbb{E}(|X^{k-1}|) < \infty$ if k is odd*
- *if $\mathbb{E}(|X^k|) < \infty$ then $\varphi_X(t) = \sum_{j=0}^{k} \frac{\mathbb{E}(X^j)}{j!}(it)^j + o(t^k)$, where $f(x) = o(x)$ if $f(x)/x \to 0$ for $x \to 0$*

Some useful results for convergence results

Chebychev's inequality: $\mathbb{P}(|X| > x) \leq \frac{\mathbb{E}(X^2)}{x^2}$

Markov inequality: $\mathbb{P}(|X| > x) \leq \frac{\mathbb{E}(|X|^r)}{x^r}$

Hölder's inequality: For $p, q > 1$ such that $1/p + 1/q = 1$ we have

$$\mathbb{E}(|XY|) \leq [\mathbb{E}(|X|^p)]^{1/p}[\mathbb{E}(|X|^q)]^{1/q}$$

Minkovski's inequality: For $r \geq 1$ we have

$$[\mathbb{E}(|X+Y|^r)]^{1/r} \leq [\mathbb{E}(|X|^r)]^{1/r} + [\mathbb{E}(|X|^r)]^{1/r}$$

Lemma 12 (Fatou's Lemma) *Let $X_1, X_2, \cdots$ be non-negative random variables, then $\mathbb{E}(\liminf X_n) \leq \liminf \mathbb{E}(X_n)$.*

Definition 13 (Tail events) *If $X_1, X_2, \cdots$ are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{H}_n = \sigma(X_{n+1}, X_{n+2}, \cdots)$ is the smallest σ-algebra in which all random variables $X_{n+1}, X_{n+2}, \cdots$ are measurable, then $\mathcal{H}_\infty := \cap_n \mathcal{H}_n$ is called the tail σ-algebra, and events contained in it are tail events.*

Theorem 14 (Kolmogorov's zero-one law) *If $X_1, X_2, \cdots$ are independent, then all tail events $H \subset \mathcal{H}_\infty$ satisfy either $\mathbb{P}(H) = 1$ or $\mathbb{P}(H) = 0$*

Definition 15 (Uniform integrability) *A sequence of r.v. $X_1, X_2, \cdots$ is uniformly integrable if*

$$\sup_{n\geq 1} \mathbb{E}(|X_n|\mathbb{1}(|X_n| > a)) \to 0 \qquad \textit{as } a \to \infty$$

Theorem 16 *Let X and $X_1, X_2, \cdots$ be random variables such that $X_n \xrightarrow{\mathbb{P}} X$ then the following statements are equivalent*

1. *$X_1, X_2, \cdots$ is uniformly integrable*
2. *$\mathbb{E}(|X_n|) < \infty$ for all n, $\mathbb{E}(|X|) < \infty$ and $X_n \xrightarrow{1} X$*
3. *$\mathbb{E}(|X_n|) < \infty$ and $\mathbb{E}(|X_n|) \to \mathbb{E}(|X|) < \infty$*

Martingales

Some Properties of martingales: Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}} = (\mathcal{F}_0, \mathcal{F}_1, \cdots)$.

- $\mathbb{E}(S_{n+m}|\mathcal{F}_n) = S_n$
- $\mathbb{E}(S_n) = \mathbb{E}(S_1)$
- $\mathbb{E}((S_n)^2)$ is non decreasing

Theorem 17 (Doob decomposition) *A $\underline{\mathcal{F}}$-submartingale $Y_0, Y_1, \cdots$ with finite means may be expressed in the form $Y_n = M_n + S_n$, where $M_1, M_2, \cdots$ is a $\underline{\mathcal{F}}$-martingale and S_n is $\mathcal{F}_{n-1}$ measurable for all n. This decomposition is unique.*

Lemma 18 (Doob-Kolmogorov inequality) *If $S_1, S_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$, then for all $\epsilon > 0$ we have*

$$\mathbb{P}\left(\max_{1\le k\le n} |S_k| \ge \epsilon\right) \le \epsilon^{-2}\mathbb{E}((S_n)^2).$$

Theorem 19 (Martingale convergence theorem) *If $S_1, S_2, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$ and $\mathbb{E}((S_n)^2) \nearrow M < \infty$, then there exists a random variable S such that $S_n \stackrel{a.s.}{\to} S$ and $S_n \to S$ in mean square.*

Definition 20 (Cauchy sequence) *A sequence of real numbers $x_1, x_2, \cdots$ is a Cauchy sequence if for all $\epsilon > 0$ there exists an N such that for all $n \ge m \ge N$, we have $|x_n - x_m| < \epsilon$.*
We know that a sequence is convergent if and only if it is a Cauchy sequence.

Theorem 21 *Let $S_0, S_1, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$ such that $S_0 = 0$ and $\mathbb{E}((S_n)^2) < \infty$ for all n. Define*

$$\langle S\rangle_n = \sum_{k=1}^{n} \mathbb{E}((S_k - S_{k-1})^2|\mathcal{F}_{k-1}) \qquad \textit{and} \qquad \langle S\rangle_\infty = \lim_{n\to\infty}\langle S\rangle_n.$$

Let $f \ge 1$ be a given increasing function satisfying $\int_0^\infty [f(x)]^{-2}dx < \infty$. Then,
(i) On $\{\omega : \langle S(\omega)\rangle_\infty < \infty\}$ $S_n \stackrel{a.s.}{\to} S$ for some random variable S
(ii) On $\{\omega : \langle S(\omega)\rangle_\infty = \infty\}$, $S_n/f(\langle S\rangle_n) \stackrel{a.s.}{\to} 0$

Theorem 22 (Strong Law of Large Numbers) *Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$ and define $S_0 = 0$ and $S_n = \sum_{k=1}^n (X_k - \mu)$ for $n \geq 1$. Then $\frac{S_n}{n} \xrightarrow{a.s.} 0$.*

Theorem 23 (Martingale Central Limit theorem) *$S_0, S_1, \cdots$ is a martingale with respect to $\underline{\mathcal{F}}$, with $S_0 = 0$ and $\mathbb{E}((S_n)^2) < \infty$ for all n. Assume that $n^{-1}\langle S\rangle_n \xrightarrow{\mathbb{P}} \sigma^2 > 0$ and for all $\epsilon > 0$*

$$\frac{1}{n}\sum_{k=1}^{n} \mathbb{E}((S_k - S_{k-1})^2 \mathbb{1}((S_k - S_{k-1})^2 > \epsilon n)) \to 0.$$

Then, $\frac{1}{\sqrt{n\sigma^2}} S_n \xrightarrow{d} \mathcal{N}(0,1)$

Theorem 24 (Optional stopping I) *Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$. If T is an a.s. bounded stopping time for $\underline{\mathcal{F}}$ (i.e. $\mathbb{P}(T \leq a) = 1$ for some $a \geq 0$), then $\mathbb{E}(S_T) = \mathbb{E}(S_1)$.*

Theorem 25 (Optional stopping II) *Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$ and T a stopping time for $\underline{\mathcal{F}}$. Then $\mathbb{E}(S_T) = \mathbb{E}(S_1)$, if the following conditions hold*

- $\mathbb{P}(T < \infty) = 1$,
- $\mathbb{E}(|S_T|) < \infty$,
- $\mathbb{E}(S_n \mathbb{1}(T > n)) \to 0$ *as* $n \to \infty$.

Theorem 26 (Optional Stopping III) *Let $S_1, S_2, \cdots$ be a martingale with respect to $\underline{\mathcal{F}}$ and T a stopping time for $\underline{\mathcal{F}}$. Then $\mathbb{E}(S_T) = \mathbb{E}(S_1)$, if the following conditions hold*

- $\mathbb{E}(T) < \infty$,
- $\mathbb{E}(|S_{n+1} - S_n||\mathcal{F}_n) \leq K$ *for all* $n < T$ *and some* $K > 0$

Wald's equation and identity: If $X_1, X_2, \cdots$ are i.i.d. random variables with $\mathbb{E}(X_1) = \mu < \infty$ and $S_n = \sum_{k=1}^n X_k$ and T is a stopping time satisfying $\mathbb{E}(T) < \infty$, then $\mathbb{E}(S_T) = \mu\mathbb{E}(T)$.
If in addition there exists a $h > 0$ such that $M(t) = \mathbb{E}(e^{tX_1}) < \infty$ for all $|t| < h$ and $M(t) > 1$ and $|S_n| < C$ for some constant $C > 0$ and all $n \leq T$, then $\mathbb{E}(e^{tS_T}[M(t)]^{-T}) = 1$.