Solutions second exam Probability III

November 25, 2020

Problem 1
(**a**) Provide the definition of a σ-algebra defined on the sample space Ω.(3p)

Solution: A collection of subsets of Ω (say $\mathcal{F}$) is s σ-algebra if 1) $\Omega \in \mathcal{F}$, 2) If $A \in \mathcal{F}$ then also the complement $A^c \in \mathcal{F}$ and 3) if $A_1, A_2, \cdots$ is a sequence of elements of $\mathcal{F}$ then also $\cup_n A_n \in \mathcal{F}$.

(**b**) Let $\mathcal{F}$ be the smallest σ-algebra on $\Omega = (0,1)$ containing all open intervals contained in $(0,1)$. Show that every single point $x \in (0,1)$ is an element of $\mathcal{F}$. (4p)

Solution: Both $(0,x) \in \mathcal{F}$ and $(x,1) \in \mathcal{F}$ because they are open intervals. Therefore, by the third defining property of a σ-algebra, $B = (0,x) \cup (x,1) \in \mathcal{F}$ and by the second defining property of a σ-algebra $B^c = \{x\} \in \mathcal{F}$.

(**c**) Let $\mathcal{G}$ be the smallest σ-algebra on $\Omega = (0,1)$ containing all single points in $(0,1)$. Show that $\mathcal{G} \neq \mathcal{F}$. That is, give an element of $\mathcal{F}$ which is not in $\mathcal{G}$ and explain why this element is not in $\mathcal{G}$. (5p)

Solution: The σ-algebra generated by single points contains all countable unions of single points and complements of those sets. Because countable unions of countable sets are countable and a union containing at least one set with a countable complement, has at most a countable complement, we have that $\mathcal{G}$ only contains sets which have countably many elements or a countable complement. the interval $(0,1/2)$ is therefore not in $\mathcal{G}$ because it has uncountably many elements and its complement $[1/2,1)$ is also uncountable.

Problem 2

a) Let X be an exponential distributed random variable with expectation $1/\lambda$. That is, $\mathbb{P}(X > x) = e^{-\lambda x}$ for all $x \geq 0$. Provide the characteristic function $\varphi_X(t)$ of X. (3p)

Hint: You may assume without proof that $\varphi_X(t) = \psi(it)$, where $i = \sqrt{-1}$ and $\psi(t)$ is the moment generating function of X.

Solution: The exponential distributed random variable has density function $f(x) = \lambda e^{-\lambda x}$ for $x > 0$. Therefore, for $t < \lambda$.

$$\psi(t) = \int_0^\infty \lambda e^{-\lambda x} e^{tx} dx = \frac{\lambda}{\lambda - t}$$

and $\varphi_X(t) = \frac{\lambda}{\lambda - it}$.

b) Let Y be a random variable with distribution defined through

$$\mathbb{P}(Y = 1) = \mathbb{P}(Y = -1) = 1/2.$$

Let $Z = YX$. Deduce what $\varphi_Z(t)$, the characteristic function of Z, is. (3p)

Solution: The quick way is to note that

$$\mathbb{E}[e^{itZ}] = \mathbb{E}[\mathbb{E}[e^{itXY}|Y]] = \frac{1}{2}\left(\mathbb{E}[e^{itX} + e^{-itX}\right) = \frac{1}{2}\left(\frac{\lambda}{\lambda - it} + \frac{\lambda}{\lambda + it}\right) = \frac{\lambda^2}{\lambda^2 + t^2}.$$

If you are less comfortable with complex numbers: Note that $\varphi_Z(t) = \mathbb{E}[\cos(tZ)] + i\mathbb{E}[\sin(tZ)]$ and using the telescoping property of expectations.

$$\mathbb{E}[\cos(tZ)] = \mathbb{E}[\cos(tXY)] = \mathbb{E}[\mathbb{E}[\cos(tXY)|Y]] = \frac{1}{2}\mathbb{E}[\cos(tX)] + \frac{1}{2}\mathbb{E}[\cos(-tX)].$$

Because $\cos(\cdot)$ is symmetric, we have that $\mathbb{E}[\cos(tX)] = \mathbb{E}[\cos(-tX)]$ and $\mathbb{E}[\cos(tZ)] = \mathbb{E}[\cos(tX)]$.

Similarly, by the anti-symmetry of $\sin(\cdot)$, we have

$$\mathbb{E}[\sin(tZ)] = \frac{1}{2}\mathbb{E}[\sin -tX)] + \frac{1}{2}\mathbb{E}[\sin(-tX)] = \frac{1}{2}\mathbb{E}[\sin -tX)] - \frac{1}{2}\mathbb{E}[\sin -tX)] = 0.$$

So, $\varphi_Z(t) = \mathbb{E}[\cos(tX)] = \frac{\mathbb{E}[e^{itX}] + \mathbb{E}[e^{i(-t)X}]}{2}$. Using part a) We then obtain that

$$\varphi_Z(t) = \frac{1}{2}\left(\frac{\lambda}{\lambda - it} + \frac{\lambda}{\lambda + it}\right) = \frac{\lambda}{2}\left(\frac{(\lambda + it) + (\lambda - it)}{(\lambda - it)(\lambda + it)}\right) = \frac{\lambda^2}{\lambda^2 + t^2}.$$

Let $Y_1, Y_2, \cdots$ and $X_1, X_2, \cdots$ be independent random variables, where $Y_1, Y_2, \cdots$ are all distributed as Y and for $k \in \mathbb{N}$, X_k is exponentially distributed with expectation $1/\sqrt{k}$. For $k \in \mathbb{N}$, define $Z_k = Y_k X_k$.
c) Provide the characteristic function of $W_n = \frac{1}{\sqrt{\log n}} \sum_{k=1}^n Z_k$. Use this to show that W_n converges in distribution to some random variable as $n \to \infty$. What is the distribution of this random variable? (6p)
Hint: You may use without proof that for a positive function $c(n) \geq 0$, which decreases to 0 as $n \to \infty$, we have

$$n^{-c(n)} \prod_{k=1}^{n} \left(1 + \frac{c(n)}{k}\right) \to 1.$$

Solution: Define $S_n = \sum_{k=1}^n Z_k$ and let $\varphi_{S_n}(t)$ be its characteristic function, while the characteristic function of Z_n is given by $\varphi_{Z_n}(t)$ and $\varphi_{W_n}(t)$ is defined similarly.
Because the Z_n's are all independent we have that

$$\varphi_{S_n}(t) = \prod_{k=1}^{n} \varphi_{Z_k}(t).$$

Furthermore,

$$\varphi_{W_n}(t) = \varphi_{S_n}(t/\sqrt{\log n}).$$

So,

$$\varphi_{W_n}(t) = \prod_{k=1}^{n} \varphi_{Z_k}(t/\sqrt{\log n}).$$

From part *b*) we know that $\varphi_{Z_k}(t) = \frac{k}{k+t^2}$ and thus that

$$\varphi_{W_n}(t) = \prod_{k=1}^{n} \frac{k}{k + t^2/\log n}$$

and thus that

$$\frac{1}{\varphi_{W_n}(t)} = \prod_{k=1}^{n} \left(1 + \frac{c(n)}{k}\right), \qquad \text{where } c(n) = t^2/\log n.$$

Now we can use the hint and observe that $n^{-c(n)} = e^{-c(n)\log n} = e^{-t^2}$. So,

$$e^{-t^2} \frac{1}{\varphi_{W_n}(t)} \to 1$$

and thus $\varphi_{W_n}(t) \to e^{-t^2}$. which is the characteristic function of a Normal distribution with mean 0 and variance 1/2. Because convergence of the characteristic functions implies convergence in distribution of the corresponding random variables.

Problem 3

Assume in all subproblems that $X, X_1, X_2, \cdots$ are random variables defined on the same probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$. Assume in both part a) and b) that $X_n \xrightarrow{\mathbb{P}} X$.

a) Show that there exists a non-random strictly increasing sequence of positive integers $n_1, n_2, \cdots$, such that $X_{n_k} \overset{a.s.}{\rightarrow} X$ as $k \to \infty$. (6p)

solution: We know that $\mathbb{P}(|X_n - X| > \epsilon) \to 0$, for all $\epsilon > 0$. Then choose n_k such that

$$\mathbb{P}(|X_{n_k} - X| > 1/k) < 1/k^2$$

Now we observe that for all $\epsilon > 0$ we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \mathbb{P}(|X_{n_k} - X| > \epsilon) &\leq \sum_{k=1}^{\infty} \mathbb{P}\left(|X_{n_k} - X| > \min(\frac{1}{k}, \epsilon)\right) \\
&= \sum_{k=1}^{\lfloor 1/\epsilon \rfloor} \mathbb{P}\left(|X_{n_k} - X| > \epsilon\right) + \sum_{k=\lfloor 1/\epsilon \rfloor}^{\infty} \mathbb{P}\left(|X_{n_k} - X| > \frac{1}{k}\right) \\
&\leq \sum_{k=1}^{\lfloor 1/\epsilon \rfloor} 1 + \sum_{k=\lfloor 1/\epsilon \rfloor}^{\infty} \mathbb{P}\left(|X_{n_k} - X| > \frac{1}{k}\right) \\
&\leq \frac{1}{\epsilon} + \sum_{k=1}^{\infty} \mathbb{P}\left(|X_{n_k} - X| > \frac{1}{k}\right) \leq \frac{1}{\epsilon} + \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty
\end{aligned}$$

and by the first Borel-Cantelli Lemma we obtain that $\mathbb{1}(|X_{n_k} - X| > \epsilon) \overset{a.s.}{\rightarrow} 0$ and by the hint it follows that $X_{n_k} \overset{a.s.}{\rightarrow} X$.

b) Show that if $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $g(X_n) \xrightarrow{\mathbb{P}} g(X)$.(6p)
Hint: Note that if $g : \mathbb{R} \to \mathbb{R}$ is continuous, then it is uniform continuous on any interval $[-M, M]$ with $M \in (0, \infty)$. That is, for all $\epsilon > 0$ there exist $\delta > 0$ such that $|g(x) - g(y)| < \epsilon$ if $|x - y| < \delta$ and $x, y \in [-M, M]$.

Solution: We need to prove that for all $\epsilon > 0$ and all $\epsilon_1 > 0$ there exists $N \in \mathbb{N}$ such that $\mathbb{P}(|g(X_n) - g(X)| > \epsilon) < \epsilon_1$ for all $n > N$.
Fix $\epsilon > 0$ and $\epsilon_1 > 0$ and let M be such that $\mathbb{P}(|X| > M - 1) < \epsilon_1/2$.
Let $\delta \in (0, 1)$ be such that $|g(x) - g(y)| < \epsilon$ if $|x - y| < \delta$ and $x, y \in [-M, M]$ (which is possible by the hint).
Let N be such that $\mathbb{P}(|X_n - X| \geq \delta) < \epsilon_1/2$ for all $n > N$ (which is possible by $X_n \xrightarrow{\mathbb{P}} X$).
Then for $n > N$ we have,

$$
\begin{aligned}
& \mathbb{P}(|g(X_n) - g(X)| > \epsilon) \\
= & \mathbb{P}(|g(X_n) - g(X)| > \epsilon, |X| > M - 1) + \mathbb{P}(|g(X_n) - g(X)| > \epsilon, |X| \leq M - 1) \\
\leq & \mathbb{P}(|X| > M - 1) \\
& + \mathbb{P}(|g(X_n) - g(X)| > \epsilon, |X_n - X| \geq \delta, |X| \leq M - 1) \\
& + \mathbb{P}(|g(X_n) - g(X)| > \epsilon, |X_n - X| < \delta, |X| \leq M - 1)
\end{aligned}
$$

The first summand is less than $\epsilon_1/2$ by the definition of M. The second summand is less than $\mathbb{P}(|X_n - X| \geq \delta)$, which is by the definition of N less than $\epsilon_1/2$. The third summand is 0, because if both $|X_n - X| < \delta$ and $|X| < M - 1$, then both $|X| < M$ and $|X_n| < M$ and by the definition of δ, $|g(X_n) - g(X)|$ must be less than ϵ. The result now follows.

Problem 4
Let $\lambda > 1$ be constant. For $i, j \in \mathbb{N}_0$ let X_{ij} be independent and identically distributed random variables with a Poisson distribution with expectation λ. That is,

$$\mathbb{P}(X_{ij} = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } k \in \mathbb{N}_0.$$

Define $Z_0 = 1$. For $i \in \mathbb{N}_0$ define

$$Z_{i+1} = \sum_{j=1}^{Z_i} X_{ij},$$

where we define $\sum_{j=1}^{0} = 0$.
a) Show (e.g. by induction) that $\mathbb{E}[Z_n] = \lambda^n$ and $\mathbb{E}[(Z_n)^2] = \lambda^n \frac{\lambda^{n+1}-1}{\lambda-1}$ for all $n \in \mathbb{N}$. (4p)

Solution:

$$\mathbb{E}[Z_n] = \mathbb{E}[\mathbb{E}[Z_n | Z_{n-1}]] = \lambda \mathbb{E}[Z_{n-1}] = \cdots = \lambda^n Z_0 = \lambda^n.$$

$$\begin{aligned} \mathbb{E}[(Z_n)^2] &= \mathbb{E}[\mathbb{E}[(Z_n)^2 | Z_{n-1}]] = \mathbb{E}[Z_{n-1}\lambda + (Z_{n+1}\lambda)^2 | Z_{n-1}]] \\ &= \lambda \mathbb{E}[Z_{n-1}] + \lambda^2 \mathbb{E}[(Z_{n-1})^2] = \lambda^n + \lambda^2 \mathbb{E}[(Z_{n-1})^2] \\ &= \lambda^n + \lambda^2(\lambda^{n-1} + \lambda^2 \mathbb{E}[(Z_{n-2})^2]) = \cdots = \lambda^n \sum_{k=0}^{n} \lambda^k = \lambda^n \frac{\lambda^{n+1} - 1}{\lambda - 1}. \end{aligned}$$

b) Show that

$$W_n := \frac{1}{\lambda^n} Z_n$$

converges almost surely to some random variable W. Furthermore, show that $\mathbb{P}(W < \infty) = 1$. (4p)

Solution: We use the martingale convergence theorem.
To do this we note that $\mathbb{E}[|W_n|] = 1$ for all n. Furthermore, $\mathbb{E}[W_{n+1} | Z_n] = \lambda Z_n / \lambda^{n+1} = W_n$ and finally.

$$\mathbb{E}[(W_n)^2] = \frac{1}{\lambda^{2n}} \mathbb{E}[(Z_n)^2] = \frac{\lambda - \lambda^{-n}}{\lambda - 1},$$

by part a). The right hand side increases to $\lambda/(\lambda - 1) < \infty$. So, we can use the theorem and we now that W_n converges almost surely and in mean square (and therefore in mean) to some random variable W. Because W_n converges to W in mean, we obtain that $\mathbb{E}[|W|] \leq \mathbb{E}[|W_n - W|] + \mathbb{E}[W_n] \to 0 + 1$. If $\mathbb{P}(W = \infty) > 0$ then $\mathbb{E}[W] \geq \infty \mathbb{P}(W = \infty) = \infty$, which is a contradiction.

c) Show that

$$V_n = \lambda^{-n} \sum_{i=1}^{n} Z_i$$

converges almost surely to $V = \left(\sum_{i=0}^{\infty} \lambda^{-i}\right) W = \frac{\lambda}{\lambda-1} W$. (4p)

Solution: Note

$$V_n = \sum_{i=1}^{n} \lambda^{-(n-i)}(\lambda^{-i} Z_i) = \sum_{i=1}^{n} \lambda^{-(n-i)} W_i.$$

We show that if for $\omega \in \Omega$, we have that $W_n(\omega) \to W(\omega)$ then $V_n(\omega) \to V(\omega)$. Note that $W_n(\omega) \to W(\omega)$ means that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|W_n(\omega) - W(\omega)| < \epsilon$.
Let $N_1 = N_1(\omega) \in \mathbb{N}$ be such that $|W_i(\omega) - W(\omega)| < \epsilon_1$ for all $n \geq N_1$. Then for all $n > N_1$ we have

$$|V_n(\omega) - V(\omega)| = |\sum_{i=1}^{N_1-1} \lambda^{-(n-i)}(W_i(\omega) - W(\omega)) + \sum_{i=N_1}^{n} \lambda^{-(n-i)}(W_i(\omega) - W(\omega)) - \sum_{i=n}^{\infty} \lambda^{-i} W(\omega)|.$$

By the triangle inequality we then obtain

$$|V_n(\omega) - V(\omega)| \leq \lambda^{-n} \sum_{i=1}^{N_1-1} \lambda^{i} |W_i(\omega) - W(\omega)| + \sum_{i=N_1}^{n} \lambda^{-(n-i)} \epsilon_1 + |\sum_{i=n}^{\infty} \lambda^{-i} W(\omega)|.$$

The first summand converges to 0 for fixed N_1 and ω. The second summand converges to $\epsilon_1 \sum_{j=0}^{\infty} \lambda^{-j} = \epsilon_1 \frac{\lambda}{\lambda-1}$. The third summand is equal to $\lambda^{-n} \frac{\lambda}{\lambda-1} W(\omega)$ which converges to 0 for fixed ω. So, if $W_n(\omega) \to W(\omega)$ then for each ϵ_1 there exists $N_2 > N_1$ such that for all $n > N_2$ we have $|V_n(\omega) - V(\omega)| \leq 2\epsilon_1$.

Problem 5

Let $p \in (0,1)$ be constant and $X_1, X_2, \cdots$ be independent and identically distributed random variables, satisfying

$$\mathbb{P}(X_1 = 1) = p \quad \text{and} \quad \mathbb{P}(X_1 = -1) = 1 - p.$$

For reasons of convenience set $X_0 = -1$. For $n \in \mathbb{N}$ let $K_n = 0$ if $X_n = -1$ and otherwise let K_n be the length of the sequence of consecutive $+1$'s ending at n. That is, $K_n = \min\{j \in \mathbb{N}_0; X_{n-j} = -1\}$.

Let $L \in \mathbb{N}$ be a given integer. We are interested in $T = \min\{n \in \mathbb{N}; K_n = L\}$, which is the first time a sequence of L consecutive $+1$'s appears.

To study T we can use the loss (or gain, if negative) of a casino in which the folowing happens:

1. At time 0 the loss of the casino is 0.
2. At each positive integer time point one new gambler arrives with gambling capital 1 SEK which he or she puts immediately at stake.
3. At time $k \in \mathbb{N}$, if $X_k = -1$, all gamblers which were still in the casino and arrived at time k or before, leave the casino empty handed.
4. At time $k \in \mathbb{N}$, if $X_k = 1$, all gamblers present at the casino (including the one that arrived at time k) multiply their capital instantly by a factor $1/p$, which they will put again at stake at time $k + 1$.

So, at time n (immediately after the arrival of the n-th gambler and the n-th bet) the number of gamblers in the casino is K_n.

a) Show that the loss of the casino at time n (when the new arrival and time n gambling has already occured) is given by

$$M_n = \frac{p^{-K_n} - 1}{1-p} - n$$

and show that $M_1, M_2, \cdots$ constitutes a martingale with respect to $\underline{\mathcal{F}}$, the filtration generated by $X_1, X_2, \cdots$. (4p)

Solution: The total number of newly brought in money by gamblers is n times 1 SEK. The last K_n gamblers are still in the casino. For $k \in \mathbb{N}$, if $K_n \geq k$, the gambler that entered at time $n-k+1$ has capital $(1/p)^k$ at time n. The total loss of the casino is the the total capital of the gamblers minus the ammount of money the gamblers brought in. Which is

$$\sum_{k=1}^{K_n} (1/p)^k - n = \left(\frac{(1/p)^{K_n+1} - 1}{1/p - 1} - 1 \right) - n = \frac{(1/p)^{K_n} - 1}{1-p} - n = M_n$$

.
We note that $\mathbb{E}[M_n] \leq \frac{(1/p)^n - 1}{1-p} + n < \infty$ for all n and

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= p\left(\frac{p^{-(K_n+1)} - 1}{1-p} - (n+1) \right) + (1-p)(0 - (n+1)) \\
&= \frac{p^{-K_n} - p}{1-p} - (n+1) = \frac{p^{-K_n} - 1}{1-p} - n = M_n,
\end{aligned}$$

and we showed that M_n is a martingale.
b) Compute $\mathbb{E}[T]$. You may use without proof that T is a stopping time with respect to $\underline{\mathcal{F}}$ and that $\mathbb{E}[T] < \infty$. (3p)

Solution: We note that for $K_n < L$ (which is the case if $n < T$), we have $|M_{n+1} - M_n| = p^{-(K_n+1)} - 1 < p^{-L}$ if $X_{n+1} = 1$ and we have

$$|M_{n+1} - M_n| = \frac{p^{-K_n} - 1}{1-p} + 1 = \frac{p}{1-p}(p^{-(K_n+1)} - 1) < \frac{p}{1-p}p^{-(L)}$$

if $X_{n+1} = -1$. So, we can use Optimal Stopping theorem III (Thm 26 of cheat sheet) and obtain that $\mathbb{E}[M_T] = M_0$ That is $\frac{p^{-L}-1}{1-p} - \mathbb{E}[T] = 0$.

c) Let $s \in [0,1]$ be a constant. Compute $\mathbb{E}[s^T]$. (5p)
Hint:A way to solve this might be to adapt step 2 above in such a way that for all $n \in \mathbb{N}$, the customer arriving at time n has initial gambling capital s^{n-1}. Note that partial answers might be worth points.

Solution: If the gambler entering at time n comes in with capital s^n, then the total capital brought in by gamblers up to and including time n is

$$\sum_{k=1}^{n} s^k = s\frac{1-s^n}{1-s}.$$

While if $k \leq K_n$ the gambler that entered at time $n-k+1$ is given by $s^{n-k+1}p^{-k}$. So, we can define the martingale

$$\begin{aligned} Y_n &= \sum_{k=1}^{K_n} s^{n-k+1}p^{-k} - s\frac{1-s^n}{1-s} = s^n p^{-1}\frac{(sp)^{-K_n}-1}{(sp)^{-1}-1} - s\frac{1-s^n}{1-s} \\ &= s\left(s^n\frac{(sp)^{-K_n}-1}{1-sp} - \frac{1-s^n}{1-s}\right) = s^n\left(\frac{s(sp)^{-K_n}-s}{1-sp} + \frac{s}{1-s}\right) - \frac{s}{1-s}. \end{aligned}$$

We note that

$$\begin{aligned} &\mathbb{E}[Y_{n+1}|\mathcal{F}_n] \\ &= sp\left(s^{n+1}\frac{(sp)^{-(K_n+1)}-1}{1-sp} - \frac{1-s^{n+1}}{1-s}\right) + s(1-p)\left(0 - \frac{1-s^{n+1}}{1-s}\right) \\ &\qquad= s^{n+1}\frac{(sp)^{-K_n}-sp}{1-sp} - s\frac{1-s^{n+1}}{1-s} \\ &\qquad= s^{n+1}\left(\frac{(sp)^{-K_n}-1}{1-sp}+1\right) - s\left(\frac{1-s^n}{1-s}+s^n\right) = Y_n. \end{aligned}$$

Similarly $\mathbb{E}[|Y_n|] < s\left(\frac{p^{-n}-s^n}{1-sp} + \frac{1-s^n}{1-s}\right) < \infty$ and from the gambler interpretation we know that the if $K_n < L$, the capital of the gamblers present changes in step $n+1$ at most

$$\sum_{k=1}^{K_n} p^{-K_n}\max((p^{-1}-1),1) + p^{-1} < \sum_{k=1}^{K_n} p^{-L} + p^{-1},$$

while the total capital brought in changes by at most s. So the chang is bounded and we can apply Thm 26 of cheat sheet. and obtain that $\mathbb{E}[Y_T] = \mathbb{E}[Y_0] = 0$ (and thus $\frac{1-s}{s}\mathbb{E}[Y_T] = 0$) , which gives

$$\mathbb{E}[s^T]\left(((sp)^{-L}-1)\frac{1-s}{1-sp}+1\right) = 1.$$