Instructions: Work alone. You are allowed to use the textbook and the class notes. You can quote results from the textbook and from the class, but state clearly which result you are using. You are **not allowed** to search the internet for solutions or hints.

Justify all your answers with a proof or a counterexample. A simple Yes or No answer, even if correct, may get partial or no credit.

Problems have multiple parts. In some cases, later parts depend on earlier ones. Even if you could not do the earlier parts, you **are allowed** to use the results of the earlier parts in the later parts.

On the first page, please have the following:

- Name
- Social security number
- Write out and sign the following pledge: *On my honor as a student I have not received help or used inappropriate resources on this exam.*
- List the problems that you have attempted.

Start each problem on a new page (but it is not necessary to start each part of a problem on a new page). Write at the top of each page which problem it belongs to.

1. Let $X = [0, 2]$. Define a topology on X as follows: A set $U \subset X$ is open in X, if $U = \emptyset$ or if $[0, 1] \subset U$ (you do not need to prove that this really is a topology on X). Determine, with justification, whether X is

   (a) [2 pts] compact.

   Solution: No. For every $x \in X\setminus[0, 1]$, let $U_x = [0, 1]\cup\{x\}$. The collections of sets $\{U_x\}_{x\in X\setminus[0,1]}$ is an infinite open cover of X, that clearly does not have any subcovers.

   (b) [1 pt] connected.

   Solution: Yes. Any two non-empty open subsets of X have a non-empty intersection, which implies that X does not have a separation.

   (c) [1 pt] Hausdorff.

   Solution: No. Points in $[0, 1]$ can not be separated by disjoint open neighborhoods.

   (d) [1 pt] metrizable.

   Solution: No, because X is not Hausdorff.

   (e) [2 pts] first countable.

   Solution: Yes. Every point $x \in X$ has a local basis consisting of *one* open neighborhood. For $x \in [0, 1]$ the set $[0, 1]$ forms a local basis. For $x \in X \setminus \{x\}$, the set $\{x\}$ forms a local basis.

   (f) [2 pts] second countable.

   Solution: No. For every $x \in X \setminus [0, 1]$, the set $\{x\}$ must be an element of every basis of this topology. So every basis has uncountably many elements.

2. Let X be a topological space, and $A \subset X$ a subspace. Define the (not necessarily continuous) function $\chi_A \colon X \to \mathbb{R}$ as follows

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

(a) [2 pts] Let $x \in X$. Prove that χ_A is continuous at x if and only if $x \notin \partial A$. (To fix a slight misstatement in the exam, a function f is said to be continuous at a point x if for every open neighborhood V of $f(x)$, $f^{-1}(V)$ contains an open neighborhood of x).

Solution: Suppose first that $x \notin \partial A$. Then either $x \in \operatorname{int}(A)$ or $x \in \operatorname{int}(X \setminus A)$. In either case, x is an element of an open set, let's call it U, on which χ_A is constant, and therefore continuous. It follows that for any open neighborhood V of $\chi_A(x)$, $\chi_A^{-1}(V) \cap U$ is an open subset of U. But U us open in X, so $\chi_A^{-1}(V) \cap U$ is an open subset of X. But this means that $\chi_A^{-1}(V)$ contains an open neighborhood of x, so χ_A is continuous at x.

Now suppose that $x \in \partial A$. We want to prove that χ_A is not continuous at x. By assumption, $x \in \overline{A} \cap \overline{X \setminus A}$. On the other hand, x is in A or in $X \setminus A$, but not both. Let us suppose that $x \in A$. Then $\chi_A(x) = 1$. Let $V = (1/2, 3/2)$. Then $\chi_A^{-1}(V) = A$. But $x \in \overline{X \setminus A}$, so every open neighborhood U of x contains a point in $X \setminus A$. In other words A, also known as $\chi_A^{-1}(V)$, does not contain any open neighborhood of x. So χ_A is not continuous at x. The case $x \in X \setminus A$ is done similarly.

(b) [2 pts] Prove that χ_A is a continuous function if and only if A is both open and closed.

Solution: By part (a) χ_A is continuous if and only if $\partial A = \emptyset$. In other words, if and only if $\overline{A} \cap \overline{X \setminus A} = \emptyset$. Suppose this holds. Since every point of X is either in A or in $X \setminus A$, it follows immediately that $A = \overline{A}$ and $X \setminus A = \overline{X \setminus A}$. In this case A and $X \setminus A$ are both closed, and therefore also both open. This proves the only if direction. For the if direction, suppose A is both open and closed. Then it follows that $\partial A = \overline{A} \cap \overline{X \setminus A} = A \cap (X \setminus A) = \emptyset$, and so χ_A is continuous.

3. [4 pts] Let $\mathbb{R}_l$ be the real line, with the topology generated by the basis of sets of the form $[a, b)$. Let

$$X = [2, 3] \cup \{-\frac{1}{n} \mid n = 1, 2, 3, \ldots\}.$$

Find the interior and the closure of X in $\mathbb{R}_l$.

Solution: First we claim that the interior of X is $[2, 3)$. First of all, it is clear that $[2, 3)$ is open in $\mathbb{R}_l$, and is a subset of X. It follows that $[2, 3)$ is contained in the interior of X. To prove that $[2, 3)$ equals the interior of X, we need to show that none of the points of $X \setminus [2, 3)$ is an interior point of X. It is clear that 3 is not an interior point, because every open neighborhood of 3 will contain points greater than 3, that are not in X. Similarly, none of the points $-\frac{1}{n}$ is an interior point, because every open neighborhood of $-\frac{1}{n}$ will contain points $-\frac{1}{n} < y < -\frac{1}{n+1}$ that are not in X.

Second, we claim that X is closed, and therefore $\overline{X} = X$. To prove that X is closed, we need to prove that the complement of X is open. But the complement of X equals to the following union

$$(-\infty, -1) \cup \bigcup_{n=1}^{\infty} (-\frac{1}{n}, -\frac{1}{n+1}) \cup [0, 1) \cup (3, \infty).$$

It is easy to see that each one of these sets is open in $\mathbb{R}_l$, and therefore their union is open in $\mathbb{R}_l$.

4. [5 pts] Let X be a topological space satisfying the following property: For every two points $a, b \in X$, there exists a finite sequence of connected subspaces $C_1, \ldots, C_n$ of X such that $a \in C_1$, $b \in C_n$ and $C_i \cap C_{i+1} \neq \emptyset$ for all $i = 1, \ldots, n-1$. Prove that X is connected.

   Solution: First, let us prove that for $C_1, \ldots, C_n$ as specified in the problem, the union $C_1 \cup \ldots \cup C_n$ is always connected. Suppose U, V are disjoint open subsets of $C_1 \cup \ldots \cup C_n$, whose union is $C_1 \cup \ldots \cup C_n$. Since C_i are connected, each C_i is either a subset of U or of V. Without loss of generality $C_1 \subset U$. But then C_2 has at least one point in U, and therefore $C_2 \subset U$. By induction, $C_i \subset U$ for all i, and $V = \emptyset$. This proves that $C_1 \cup \ldots \cup C_n$ is connected.

   Now we know that for every two points $a, b \in X$, there exists a connected subset of X that contains both a and b. It follows that every two points of X are in the same connected component of X. So X has one connected component, which means that X is connected.

5. Let S^2 be a sphere, and let $a, b \in S^2$ be two distinct points. Let $S^2/\{a, b\}$ be the quotient space of S^2 by the relation that identifies a and b.

   (a) [3 pts] Describe a CW structure on $S^2/\{a, b\}$. Specify clearly how many cells of each dimension there are.

   Solution: Choose a circle passing through a, b that separates S^2 in two components. For example it could be the great circle through a and b. S^2 has a cell structure, with

   - two zero-cells: a and b
   - two one-cells: the two arcs of the circle connecting a and b.
   - two two-cells: the two hemispheres of S^2 intersecting at the circle.

   It follows that $S^2/\{a, b\}$ has a cell structure with a single one-cell - the class $[a, b]$, two one-cells and two two-cells.

   (b) [1 pt] Compute the Euler characteristic of $S^2/\{a, b\}$.
   Solution: The Euler characteristic is $1 - 2 + 2 = 1$.

6. Let S^2 be the two-dimensional sphere and let T be the torus. Let $x_1, x_2, x_3 \in S^2$ be three distinct points. Let $S^2 \setminus \{x_1, x_2, x_3\}$ denote the complement of three points in S^2 (do not confuse it with the quotient space of the previous question).

   (a) [2 pts] Use the van Kampen theorem to prove that $\pi_1(S^2 \setminus \{x_1, x_2, x_3\})$ is isomorphic to the free group on 2 generators.

   Solution: We know that $S^2 \setminus \{x_1\}$ is homeomorphic to $\mathbb{R}^2$. Therefore $S^2 \setminus \{x_1, x_2, x_3\}$ is homeomorphic to the complement of two points in the plane. Without loss of generality we may take these points to be $(-2, 0)$ and $(0, 2)$. So we need to apply the SVK theorem to $\mathbb{R}^2 \setminus \{(-2, 0), (0, 2)\}$.

   Let $U = \{(x, y) \in \mathbb{R}^2 \setminus \{(-2, 0), (0, 2)\} \mid x < 1\}$ and $V = \{(x, y) \in \mathbb{R}^2 \setminus \{(-2, 0), (0, 2)\} \mid x > -1\}$. Then $\mathbb{R}^2 \setminus \{(-2, 0), (0, 2)\} = U \cup V$, $U \cap V = (-1, 1) \times \mathbb{R}$. It is easy to see that U and V is each homeomorphic to $\mathbb{R}^2 \setminus \{pt\}$, and so it homotopy equivalent to S^1. The intersection $U \cap V$ is contractible, in particular simply connected. By the SVK theorem, $\pi_1(S^2 \setminus \{x_1, x_2, x_3\}) \cong \pi_1(S^1) * \pi_1(S^1) = F_2$.

(b) [2 pts] Let $y \in T$ be any point. Prove that the spaces $T \setminus \{y\}$ and $S^2 \setminus \{x_1, x_2, x_3\}$ both contain the wedge sum (i.e., one-point union) $S^1 \vee S^1$ as a deformation retract. Conclude that they are homotopy equivalent.

Solution: T is obtained by attaching a single two-dimensional cell to $S^1 \vee S^1$. It follows that $T \setminus \{y\}$ deformation retracts onto $S^1 \vee S^1$.

As to $S^2 \setminus \{x_1, x_2, x_3\}$, once again, this is homeomorphic to $\mathbb{R}^2$ minus two points. It is not hard to show that this space deformation retracts onto $S^1 \vee S^1$. Details are left to the reader.

(c) [1 pt] The Jordan curve theorem asserts that the complement of any closed non-self-intersecting loop in $\mathbb{R}^2$ has two connected components. Assuming the Jordan curve theorem, prove that the spaces $T \setminus \{y\}$ and $S^2 \setminus \{x_1, x_2, x_3\}$ are not homeomorphic.

Solution: The point is that there is an embedded circle $S^1 \subset T \setminus \{y\}$, whose complement is path-connected. Indeed, choose a circle of the form $S^1 \times \{*\}$ that does not pass through $\{y\}$. Then

$$T \setminus (S^1 \times \{*\}) = (S^1 \times S^1) \setminus (S^1 \times \{*\}) \cong S^1 \times (0, 1),$$

which is clearly a connected surface. But then $T \setminus \{y\} \cong S^1 \times (0, 1) \setminus \{y\}$ is still connected, because a connected surface remains connected after removing a point.

Now suppose that we have a homeomorphism $f \colon T \setminus \{y\} \xrightarrow{\cong} S^2 \setminus \{x_1, x_2, x_3\}$. Then $f(S^1 \times \{x\})$ is an embedded circle in $S^2 \setminus \{x_1, x_2, x_3\}$, and also in $S^1 \setminus \{x_1\} \cong \mathbb{R}^2$. f induces a homeomorphism

$$(T \setminus \{y\}) \setminus S^1 \times \{x\} \cong (S^2 \setminus \{x_1, x_2, x_3\}) \setminus f(S^1 \times \{x\}).$$

By the Jordan curve theorem, the complement the circle $f(S^1 \times \{*\})$ in $S^1 \setminus \{x_1\}$ is not connected. But then it follows that the complement of this circle in $S^2 \setminus \{x_1, x_2, x_3\}$ is also not connected (why?). So f induces a homeomorphism between a connected space and a non-connected space, which is impossible.