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MA 5020 - Abstract Algebra
Final Exam

Nov 30, 2021

Instructions: Textbooks, notes and calculators are not allowed. You may quote results that you learned during the class. When you do, state precisely the result that you are using. Unless explicitly instructed otherwise, be sure to justify your answers, and show clearly all steps of your solutions. In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

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1. For each of the following statements, determine if it is (always) true or (sometimes) false. Justify your answers either by a brief and clear argument or by a counterexample.
   (a) [2 pts] Suppose G is a group, with subgroups G_1, G_2 and H. If $H \subset G_1 \cup G_2$ then either $H \subset G_1$ or $H \subset G_2$.
   (b) [2 pts] Let G_1, G_2 be groups. For every subgroup $K \subset G_1 \times G_2$, there exist subgroups $H_1 \subset G_1$ and $H_2 \subset G_2$ such that $K = H_1 \times H_2$.
   (c) [2 pts] Let R, S be commutative rings with unit. For every ideal I of $R \times S$, there exists ideals I_1 of R and I_2 of S such that $I = I_1 \times I_2$.

2. Let S_{10} be the group of permutations of a set with 10 elements.
   (a) [2 pts] Does S_{10} have a cyclic subgroup of order 30?
   (b) [2 pts] How many elements are there in S_{10} that commute with $(12)(34)(567)$?

3. (a) [3 pts] Let G be a finite group, and H a proper subgroup of G. Prove that there exists an element $g \in G$ that is not conjugate to any element of H.
   (b) [3 pts] Suppose that G is a finite group acting transitively on a set X[1]. Prove that there exists an element $g \in G$ such that $gx \neq x$ for all $x \in X$.
   (c) [2 pts] (optional bonus problem!) Show that Part (a) may not hold if G is an infinite group.

4. [4 pts] Prove that a group of order 56 has a normal p-Sylow subgroup for some prime p.

5. [5 pts] Let R be a commutative ring with a unit element $1 \neq 0$. Suppose that $R[x]$ is a principal ideal domain (I.e., an integral domain, where every ideal is principal). Prove that R is a field.

6. In this question, all coefficients are taken to be in $\mathbb{Z}/5$. Let $a \in \mathbb{Z}/5$. Consider the polynomial, depending on a, $p(x) = x^2 + ax + 1$. As usual, let (p) be the ideal of $\mathbb{Z}/5[x]$ generated by $p(x)$.
   (a) [2 pts] How many elements are there in the quotient ring $\mathbb{Z}/5[x]/(p)$? If the answer depends on a, show explicitly how it depends. If it does not depend, explain why.
   (b) [2 pts] For which values of a, is the polynomial $p(x) = x^2 + ax + 1$ irreducible?
   (c) [1 pt] For which values of a, is the quotient ring $\mathbb{Z}/5[x]/(p)$ a field?

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[1] Recall that an action is transitive if for every $x, y \in X$ there exists a $g \in G$ such that $y = gx$