MM7041–Topology
Final Exam

Instructions: Work alone. You are not allowed to use the textbook and the class notes. You can quote results that you learned in the class. Be sure to state clearly what results you are using.

Justify all your answers with a proof or a counterexample. A simple Yes or No answer, even if correct, may get partial or no credit.

Problems have multiple parts. In some cases, later parts depend on earlier ones. Even if you could not do the earlier parts, you **are allowed** to use the results of the earlier parts in the later parts.

On the first page, please have the following:

- Name
- Social security number
- List the problems that you have attempted.

Start each problem on a new page (but it is not necessary to start each part of a problem on a new page). Write at the top of each page which problem it belongs to.

1. Let $A \subset \mathbb{R}^n$ be a subspace. We say that A *has property* (B) if every continuous map $f\colon A \to \mathbb{R}$ is bounded. For each of the following statements, determine if it is true or false (with a proof or a counterexample, of course).
   (a) [2 pts] If A is compact then it has property (B).
   (b) [2 pts] If A has property (B) then it is compact.
2. Let X and Y be topological spaces and let $p_Y\colon X \times Y \to Y$ be a projection map.
   (a) [2 pts] Is p_Y always an open map, for all spaces X and Y?
   (b) [2 pts] Prove that if X is compact, then p_Y is a closed map.
   (c) [2 pts] Is p_Y a closed map for *all* spaces X and Y?
3. [4 pts] Let X be the quotient space of $\mathbb{R}^2$ by the relation $(x, y) \sim (-x, -y)$, for all $(x, y) \in \mathbb{R}^2$. Prove that X and $\mathbb{R}^2$ are homeomorphic.

   Suggestion: use the complex square function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. You may take for granted that f is continuous.
4. Let $X = [0, 1] \times [0, 1]$. Let $a = (0, 0)$, $b = (0, 0.5)$ and $c = (0.5, 0.5)$. For each of the following statements, determine if it is true or false, and give a proof or a counterexample.
   (a) [2 pts] There is a homeomorphism $X \setminus \{a\} \cong X \setminus \{b\}$.
   (b) [2 pts] There is a homeomorphism $X \setminus \{b\} \cong X \setminus \{c\}$.
   (c) [2 pts] There is a homeomorphism $X \setminus \{a\} \cong X \setminus \{c\}$.

5. [4 pts] Let

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

denote, as usual, the unit sphere in $\mathbb{R}^3$. Let X be the quotient space of S^2 by the relation $(x, y, 0) \sim (-x, -y, 0)$ whenever $(x, y, 0) \in \mathbb{R}^3$. Calculate the fundamental group of X.

6. Let E_1, E_2, E_3 and E_4 be connected and locally path-connected spaces. Suppose that $\pi_1(E_1) \cong \mathbb{Z}$, $\pi_1(E_2) \cong \mathbb{Z}/2$, $\pi_1(E_3) \cong \mathbb{Z}/4$, $\pi_1(E_4) \cong \Sigma_3$ (where Σ_3 denotes the symmetric group on three elements).

   (a) [2 pts] Find all the ordered pairs of distinct indices $1 \leq i \neq j \leq 4$ for which there *might* be a covering map $E_i \to E_j$, based on the information you are given.

   (b) [2 pts] For each of the possible covering maps that you included in part (a), find the size of the fiber.

   (If you are not sure about the answer to part (a), you are welcome to explain how you would solve part (b) if you knew the answer to (a).)

   (c) [2 pts] Which of the coverings of part (a) are normal? (same comment as in part (b) applies here)