Instructions: Work alone. You are not allowed to use the textbook and the class notes. You can quote results that you learned in the class. Be sure to state clearly what results you are using.

Justify all your answers with a proof or a counterexample. A simple Yes or No answer, even if correct, may get partial or no credit.

Problems have multiple parts. In some cases, later parts depend on earlier ones. Even if you could not do the earlier parts, you **are allowed** to use the results of the earlier parts in the later parts.

1. Let $A \subset \mathbb{R}^n$ be a subspace. We say that A *has property* (B) if every continuous map $f\colon A \to \mathbb{R}$ is bounded. For each of the following statements, determine if it is true or false (with a proof or a counterexample, of course).
   (a) [2 pts] If A is compact then it has property (B).
   Solution: True. Proof: Suppose A is compact, and $f\colon A \to \mathbb{R}$ is a continuous map. Then $f(A)$ is compact, and therefore bounded.
   (b) [2 pts] If A has property (B) then it is compact.
   Solution: True. Proof: We will prove the contrapositive. Suppose A is not compact. We will prove that there exists an unbounded map $f\colon A \to \mathbb{R}$. Since A is not compact, we know that A is either not bounded or not closed (or both). Suppose first that A is not bounded. Then the function $f(\mathbf{x}) = ||\mathbf{x}||$ is an unbounded map from A to $\mathbb{R}$. Now suppose that A is not closed. Then A has a limit point $\mathbf{z}$ that is not an element of A. The function $f(\mathbf{x}) = \frac{1}{||\mathbf{x}-\mathbf{z}||}$ defines an unbounded map from A to $\mathbb{R}$.

2. Let X and Y be topological spaces and let $p_Y\colon X \times Y \to Y$ be the projection map.
   (a) [2 pts] Is p_Y always an open map, for all spaces X and Y?
   Solution: Yes. Proof: since direct image preserves unions of sets, it is enough to prove that p_Y takes elements of a basis of the product topology to open sets. The standard basis for $X \times Y$ consists of sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y. It is clear that $p_Y(U \times V) = U$, so p_Y is open.
   (b) [2 pts] Prove that if X is compact, then p_Y is a closed map.
   Solution: Let $C \subset X \times Y$ be a closed subset. We want to prove that $p_Y(C)$ is closed in X. We will prove that the complement $X \setminus p_Y(C)$ is open. Let $x \in X \setminus p_Y(C)$. It follows that $C \cap \{x\} \times Y = \emptyset$. Or equivalently, $\{x\} \times Y \subset (X \times Y \setminus C)$. By the tube lemma, there exists an open neighborhood $x \in U \subset X$ such that $U \times Y \subset (X \times Y \setminus C)$. It follows that $U \subset X \setminus p_Y(C)$. We have proved that every point of $X \setminus p_Y(C)$ has an open neighborhood in X that is contained in $X \setminus p_Y(C)$, so $X \setminus p_Y(C)$ is open.
   (c) [2 pts] Is p_Y a closed map for *all* spaces X and Y?
   Solution: No. For example the set $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ is a closed subset of $\mathbb{R}^2$, but its projection onto the x-axis is not closed.

3. [4 pts] Let X be the quotient space of $\mathbb{R}^2$ by the relation $(x, y) \sim (-x, -y)$, for all $(x, y) \in \mathbb{R}^2$. Prove that X and $\mathbb{R}^2$ are homeomorphic.

Suggestion: use the complex square function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. You may take for granted that f is continuous.

Solution: Consider the complex function $f(z) = z^2$. Clearly f satisfies $f(z) = f(-z)$. Therefore f induces a continuous map $\bar{f}\colon X \to \mathbb{C}$. We need to prove that $\bar{f}$ is a homeomorphism.

First, we prove that $\bar{f}$ is injective. Suppose that for some complex numbers z and w, $f(z) = f(w)$. Then obviously $z = \pm w$. So $f(z) = f(w)$ **if and only if** z and w have the same image in X. It follows that the induced map $\bar{f}\colon X \to \mathbb{C}$ is injective.

Since every complex number has a square root (most have two), f is surjective and therefore $\bar{f}$ is also surjective.

Finally it remains to prove that $\bar{f}$ is a homeomorphism. It is enough to prove that $\bar{f}$ is either open or closed, and for this it is enough to prove that the squaring map $f\colon \mathbb{C} \to \mathbb{C}$ is either open or closed. In fact, f is both open and closed, and proof of either property would suffice to answer the question. We will prove that f is closed.

Proof that f is a closed map: We will use the following property of f. The preimage of a bounded set under f is bounded. Indeed, suppose that z is a complex number and C is a positive real such that $|z| < C$. Suppose $w^2 = z$. Then $|w| < \sqrt{C}$. Thus the preimage of the set $\{z \mid |z| < C\}$ is contained in disc of radius $\sqrt{C}$. It follows that the preimage of every bounded set is bounded.

Now let $A \subset \mathbb{C}$ be a closed set. We want to prove that $f(A)$ is closed. Let D_R be the closed disc about zero of radius R. We claim that it is enough to prove that $f(A) \cap D_R$ is closed for every positive number R. Indeed, Suppose $W \subset \mathbb{C}$ satisfies the property that $W \cap D_R$ is closed for every R. Let us prove that every limit point of W is in W. Let u be a limit point of W. This means that every open neighborhood of u contains a point of $W \setminus \{u\}$. Choose an $R > |u|$. Then every open neighborhood of u contains a point of $W \cap D_R$ different from u, and therefore u is a limit point of $W \cap D_R$. We assume that $W \cap D_R$ is closed, so $u \in W \cap D_R \subset W$.

It remains to prove that $f(A) \cap D_R$ is closed for every positive number R. But then

$$f(A) \cap D_R = f(A \cap f^{-1}(D_R)).$$

We have seen that the set $f^{-1}(D_R)$ is bounded. It is also closed, since D_R is closed and f is continuous. It follows that $f^{-1}(D_R)$ is compact, and therefore $A \cap f^{-1}(D_R)$ is compact, and therefore $f(A \cap f^{-1}(D_R))$ is closed.

4. Let $X = [0, 1] \times [0, 1]$. Let $a = (0, 0)$, $b = (0, 0.5)$ and $c = (0.5, 0.5)$. For each of the following statements, determine if it is true or false, and give a proof or a counterexample.

   (a) [2 pts] There is a homeomorphism $X \setminus \{a\} \cong X \setminus \{b\}$.

   Solution: True. The proof follows from the following three rather obvious assertions.

   1. The points a and b lie on the boundary of X.
   2. There is a homeomorphism of X with the closed disc D^2, which takes boundary to boundary.
   3. For any two points p, q on the boundary of D^2, the spaces $D^2 \setminus \{p\}$ and $D^2 \setminus \{q\}$ are homeomorphic.

The proof of 1. is obvious. The proof of 3. is also obvious, using rotation symmetry of D^2. For assertion 2., it is easy enough to write an explicit homeomorphism. First, there is a homeomorphism $[0,1]\times[0,1]\cong[-1,1]\times[-1,1]$. Second, we can define a homomorphisms $g\colon[-1,1]\times[-1,1]\to D^2$ by the following formula

$$g(x,y)=\begin{cases}(0,0) & \text{if } (x,y)=(0,0)\\ \frac{\max(|x|,|y|)}{\sqrt{x^2+y^2}}(x,y) & \text{otherwise}\end{cases}$$

I leave you to check that g is indeed a homeomorphism (in particular, one has to check that g is continuous at the origin).

(b) [2 pts] There is a homeomorphism $X\setminus\{b\}\cong X\setminus\{c\}$.
Solution: False. $X\setminus\{a\}$ is contractible, while $X\setminus\{c\}$ is homotopy equivalent to S^1.

(c) [2 pts] There is a homeomorphism $X\setminus\{a\}\cong X\setminus\{c\}$.
Solution: False. Follows from (a) and (b).

5. [4 pts] Let

$$S^2=\{(x,y,z)\in\mathbb{R}^3\mid x^2+y^2+z^2=1\}$$

denote, as usual, the unit sphere in $\mathbb{R}^3$. Let X be the quotient space of S^2 by the relation $(x,y,0)\sim(-x,-y,0)$ whenever $(x,y,0)\in\mathbb{R}^3$. Calculate the fundamental group of X.

Solution: The answer is that $\pi_1(X)\cong\mathbb{Z}/2$. There are many ways to prove it. Here is one. We will defined two open subsets $U,V\subset X$, and apply the van Kampen theorem. First, we will define open subsets $\tilde U,\tilde V\subset S^2$ as follows

$$\tilde U=\{(x,y,z)\in S^2\mid z<1\}$$
$$\tilde V=\{(x,y,z)\in S^2\mid z>0\}.$$

Let U and V be the images of $\tilde U$ and $\tilde V$ under the quotient map $S^2\to X$. It is clear that $\tilde U$ and $\tilde V$ are saturated open subsets of S^2, so U and V are open subsets of X, satisfying the conditions of van Kampen. Therefore

$$\pi_1(X)\cong\pi_1(U)*_{\pi_1(U\cap V)}\pi_1(V).$$

It is easy to see that there are homeomorphisms $V\cong\mathbb{R}^2$ and $U\cap V\cong\mathbb{R}^2\setminus\{0\}$. it follows that $\pi_1(V)$ is the trivial group, and $\pi_1(U\cap V)\cong\mathbb{Z}$. In fact, the latter calculation does not matter, because of the following:

<u>claim</u>: The inclusion map $U\cap V\hookrightarrow U$ is null-homotopic (i.e., homotopic to the constant map).

<u>proof of claim</u>: Consider first the inclusion $\tilde U\cap\tilde V\hookrightarrow\tilde U$. Since $\tilde U\cong\mathbb{R}^2$, this inclusion is certainly null-homotopic. We have a diagram of the following form

$$\begin{array}{ccc}\tilde U\cap\tilde V & \xrightarrow{\cong} & U\cap V\\ \downarrow & & \downarrow\\ \tilde U & \to & U.\end{array}$$

The left map is null homotopic, therefore the composition from top left to bottom right is null-homotopic. Furthermore, the top map is a homeomorphism, and it follows that the right map is null-homotopic. □

It follows that $\pi_1(X) \cong \pi_1(U)$. It is easy to see that U deformation retracts onto the quotient of the lower hemisphere $\{(x, y, z) \in S^2 \mid z \leq 0\}$ by the relation $(x, y, 0) \sim (-x, -y, 0)$. This space is homeomorphic to $\mathbb{R}P^2$, and therefore $\pi_1(X) \cong \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$.

6. Let E_1, E_2, E_3 and E_4 be connected and locally path-connected spaces. Suppose that $\pi_1(E_1) \cong \mathbb{Z}$, $\pi_1(E_2) \cong \mathbb{Z}/2$, $\pi_1(E_3) \cong \mathbb{Z}/4$, $\pi_1(E_4) \cong \Sigma_3$ (where Σ_3 denotes the symmetric group on three elements).

   (a) [2 pts] Find all the ordered pairs of distinct indices $1 \leq i \neq j \leq 4$ for which there *might* be a covering map $E_i \to E_j$, based on the information you are given.

   Solution: A covering map always induces an injective homomorphism of fundamental groups. So there could be a covering map $E_i \to E_j$ if $\pi_1(E_i)$ is isomorphic to a subgroup of $\pi_1(E_j)$. It follows that there could be a covering map from $E_2 \to E_3$ and $E_2 \to E_4$.

   (b) [2 pts] For each of the possible covering maps that you included in part (a), find the size of the fiber.

   Solution: The size of the fiber is the index of the induced homomorphism of fundamental groups. Therefore if there is a covering map $E_2 \to E_3$, the size of its fiber is **two**. Similarly, if there is a covering map $E_2 \to E_4$, its fiber has size **three**.

   (If you are not sure about the answer to part (a), you are welcome to explain how you would solve part (b) if you knew the answer to (a).)

   (c) [2 pts] Which of the coverings of part (a) are normal? (same comment as in part (b) applies here)

   Solution: A covering $E \to B$ is normal if the image of $\pi_1(E)$ is a normal subgroup of $\pi_1(B)$. The image of $\mathbb{Z}/2$ in $\mathbb{Z}/4$ is necessarily normal, while the image of $\mathbb{Z}/2$ in Σ_3 is never normal. So the first covering map is normal, while the second one is not.