MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Tentamensomskrivning i
Combinatorics
7.5 hp
11th January 2022

Please read carefully the general instructions:

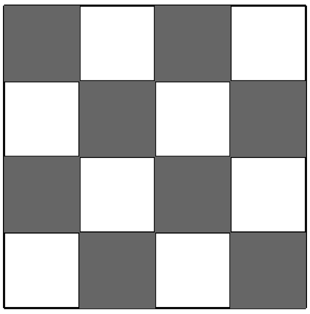
- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved. A score of at least 15 points will ensure a pass grade.

GOOD LUCK!

1. **Partitions:** (2 points)
   Consider r a positive integer and let a_r be the number of (unordered) partitions of r such that:
   - no summand is larger than 4;
   - 3 appears at least 3 times;
   - 4 appears at most 2 times.

   Express of the generating function of a_r, $r \in \mathbb{N}_{>0}$, as a quotient of polynomials.

2. **Rook polynomials:**
   (a) (2 point) Define the **rook numbers** and the **rook polynomial** of a chessboard C.
   (b) (3 points) Calculate the rook polynomial of the following 4×4 chessboard.



   (c) (2 point) State formally how the rook polynomial of the union of two disjoint chessboards C_1 and C_2 can be written in terms of the rook polynomials of the C_i's
   (d) (2 points) Prove your statement in point (d).

3. **Recursion:**
   Consider the following recursion relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 5$$

   With boundary conditions $a_0 = 0$ and $a_1 = 1$.
   (a) (3 points) Solve the relation finding a closed formula for a_n.
   (b) (2 points) Express the generating function of the sequence $\{a_n\}_{n\in\mathbb{N}}$ as a quotient of polynomials.

4. **Graphs:**
   Consider the (simple and loop-free) complete bipartite graph $K_{n,m}$. Give conditions on n and m such that
   (a) (2 points) Give conditions on n and m such that $K_{n,m}$ is connected.
   (b) (2 points) Give conditions on n and m such that $K_{n,m}$ has an Euler circuit.
   (c) (2 points) Give conditions on n and m such that $K_{n,m}$ has an Hamilton path.
   (d) (2 points) Compute the chromatic polynomial of $K_{2,2}$. (**Formula**: you can use that $p(K_n, x) = x(x-1)(x-2)\cdots(x-n)$)

5. **Latin squares:**
   Let $q = p^n$, with n a positive integer and p a prime different from 2 and 3. Define the $q \times q$ matrix $A = (a_{ij})$ by $a_{ij} \equiv 2i + j (\mod q)$
   (a) (2 points) Write A when $q = 5$. Observe that it is a Latin square.
   (b) (2 points) For $q = 5$ find a Latin square which is orthogonal to A. (**Hint:** 3 is a unit in $\mathbb{F}_5$)
   (c) (2 points) Show that for every q, the matrix A is a Latin square. (**Hint:** You need to show that $a_{ij} = a_{ik}$ implies $j = k$ and that $a_{ij} = a_{lj}$ implies $i = l$.)