Reminder on Topology

X topological space

- set together with a collection of "open sets" such that

- $\emptyset, X$ are open
- U, V open $\Rightarrow U \cap V$ open
- U_i, $i \in I$, open $\Rightarrow \bigcup_{i \in I} U_i$ open

- A map $f: X \to Y$ between topological spaces is <u>continuous</u> if

$$U \subseteq Y \text{ open} \Rightarrow f^{-1}(U) \subseteq X \text{ open}$$

· f is a homeomorphism if it is bijective and $f^{-1}: Y \to X$ is continuous.

$X \cong Y$ X, Y are "homeomorphic"

Subspaces X top. space, $A \subseteq X$ subset

$\rightsquigarrow$ subspace topology on A: $U \subseteq A$ open $\Leftrightarrow \exists V \subseteq X$ open
$U = A \cap V$

Disjoint union X, Y top. spaces

$X \sqcup Y$ disjoint union of X, Y

$U \subseteq X \sqcup Y$ open $\Leftrightarrow$ $U \cap X \subseteq X$ open and
$U \cap Y \subseteq Y$ open.

Product X, Y top. spaces

$U \subseteq X \times Y$ open $\Leftrightarrow$ $\forall (x,y) \in X \times Y$ $\exists$ open sets $W \subseteq X$, $V \subseteq Y$
$(x,y) \in W \times V \subseteq U$

<u>Quotient spaces</u> X top. space

$\sim$ an equivalence relation on X

$[x] = \{y \in X \mid x \sim y\}$ equivalence class of $x \in X$.

$X/\sim \; = \{[x] \mid x \in X\}$

$q : X \longrightarrow X/\sim$, $q(x) = [x]$

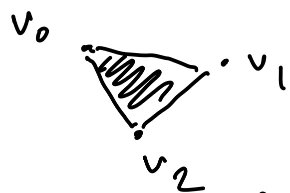
$\mathcal{U} \subseteq X/\sim$ open $\overset{\text{def.}}{\Longleftrightarrow}$ $q^{-1}(\mathcal{U}) \subseteq X$ open

Simplices

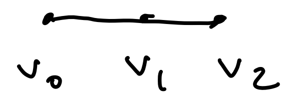
$v_0, \dots, v_n \in \mathbb{R}^m$

Assume they are in general position, i.e., $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are linearly independent

Ex:



general position



not in general position

The n-simplex spanned by $v_0, \dots, v_n$ is the space

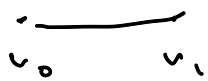
$$[v_0, \dots, v_n] = \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; 0 \le t_i \le 1,\ \sum_{i=0}^{n} t_i = 1 \right\} \subseteq \mathbb{R}^m$$

convex hull of $v_0, \dots, v_n$.

t_i "barycentric coordinates".

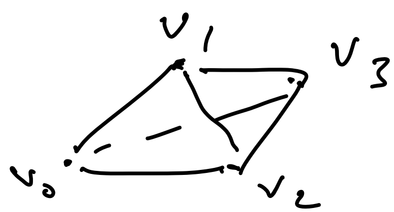
0 - simplex

1 - simplex



2 - simplex

3 - simplex



The standard n-simplex

$$\Delta^n = [e_0, \dots, e_n] = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} 0 \le t_i \le 1 \\ \sum_i t_i = 1 \end{array} \right\}$$

$e_0, \dots, e_n$ standard basis for $\mathbb{R}^{n+1}$

$$d^i : \Delta^{n-1} \to \Delta^n$$

$$d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

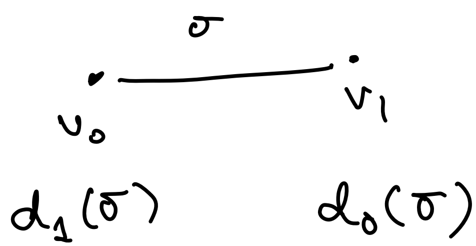
(image of d^i) $=: d_i(\Delta^n)$.

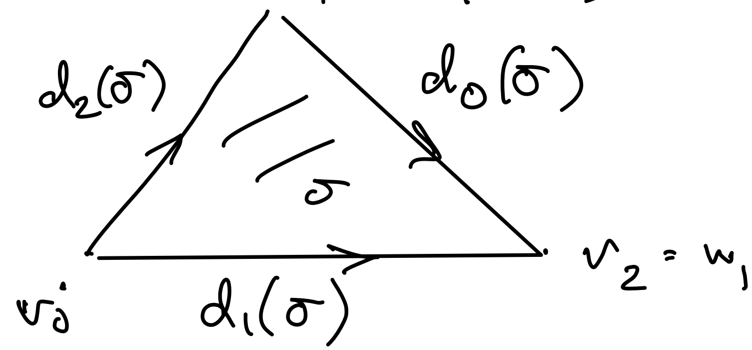
- The ith face of $[v_0, \dots, v_n] = \sigma$

$$d_i(\sigma) = [v_0, \dots, \hat{v}_i, \dots, v_n]$$
$$= [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$$

$d_i d_j = d_{j-1} d_i$
$i < j$

$$d_0 d_2(\sigma) = \overset{w_0}{v_1} = d_1 d_0(\sigma)$$





Δ - complexes

An (abstract) Δ-complex is a collection of sets $K_0, K_1, K_2, \ldots$ together with functions

$$d_i : K_n \longrightarrow K_{n-1} \qquad 0 \le i \le n$$

such that

$$d_i^{n-1} d_j^n = d_{j-1}^{n-1} d_i^n \quad \text{if } i<j$$

$$\begin{tikzcd} K_n \arrow[r, "d_j^n"] \arrow[dr, bend right, "d_i^n"'] & K_{n-1} \arrow[r, "d_i^{n-1}"] & K_{n-2} \\ & K_{n-1} \arrow[ur, "d_{j-1}^{n-1}"'] & \end{tikzcd}$$

The geometric realization of K is the topological space

$$|K| = \coprod_{n \geq 0} K_n \times \Delta^n \Big/ \sim$$

where $(\underset{\in K_{n-1}}{d_i(\sigma)}, \underset{\in \Delta^{n-1}}{t}) \sim (\underset{\in K_n}{\sigma}, \underset{\in \Delta^n}{d^i(t)})$

$\sigma \in K_n$, $t = (t_0, \dots, t_{n-1}) \in \Delta^{n-1}$

A Δ-complex structure on a top. space X is an abstract Δ-complex K with a homeomorphism $X \cong |K|$.

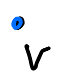
Examples

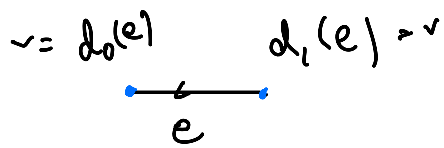
(1) $K_0 = \{v\}$ $\qquad d_0(e) = d_1(e) = v$

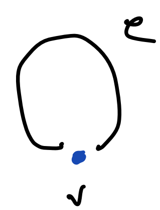
$K_1 = \{e\}$ $\qquad |K| \cong S^1$ circle

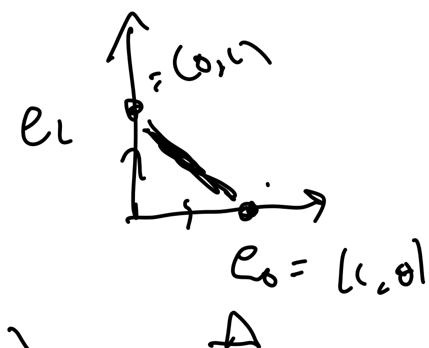
$K_n = \emptyset$, $n > 1$

$$\{v\} \times \Delta^0 \sqcup \{e\} \times \Delta^1 / \sim$$









$$d^0 \downarrow \begin{array}{l} \Delta^0 = \{1\} \subseteq \mathbb{R} \\ \Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1,\ 0 \leq t_i \leq 1\} \end{array}$$

$$d^0(1) = (0, 1)$$

$$(v, 1) = (d_0(e), 1) \sim (e, d^0(1)) = (e, (0,1))$$

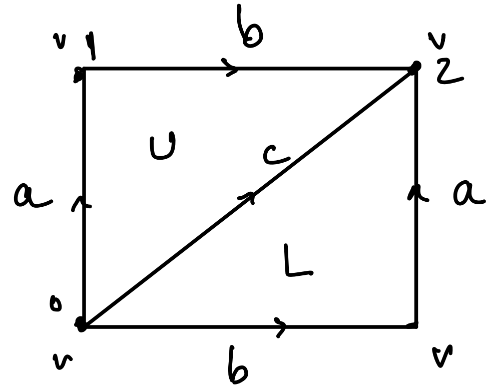
$$(v, 1) = (d_1(e), 1) \sim (e, d^1(1)) = (e, (1,0))$$

(2) $K_0 = \{v\}$

$K_1 = \{a, b, c\}$

$K_2 = \{U, L\}$

$K_n = \emptyset \quad n > 2$



$d_0(a) = d_1(a) = v$

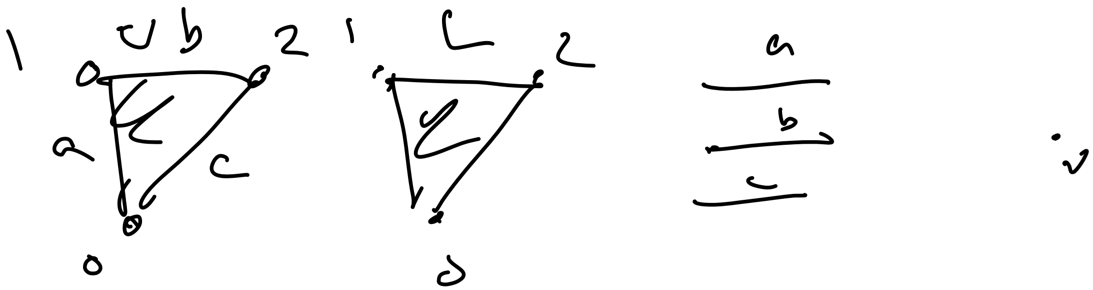
$d_0(b) = d_1(b) = v$

$d_0(c) = d_1(c) = v$

σ	d_0	d_1	d_2
U	b	c	a
L	a	c	b

$|K| \cong T^2$

torus


1
b
2
a
c
0
1
2
a
b
c

Simplicial homology

K abstract Δ-complex

$\Delta_n(K) = \mathbb{Z} K_n$ free abelian group with basis K_n

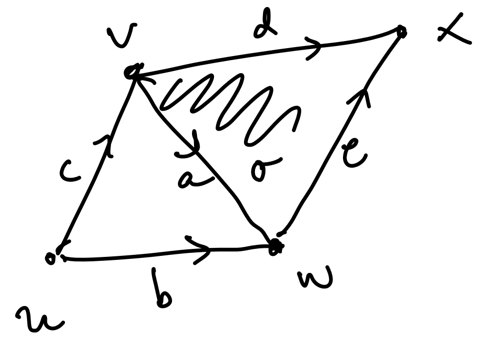
Thus, elements of $\Delta_n(K)$ are formal linear combinations

$$c = \sum_{\sigma \in K_n} n_\sigma \cdot \sigma \quad , \qquad n_\sigma \in \mathbb{Z},$$

The boundary homomorphism

$$\partial_n : \Delta_n(K) \longrightarrow \Delta_{n-1}(K)$$

$$\partial(\sigma) = \partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i d_i(\sigma)$$

Ex: K 

$a + e \in \Delta_1(K)$

$107 \cdot b - 16d \in \Delta_1(K)$

$$\partial(a+e) = \partial(a) + \partial(e) = (w-v) + (x-w)$$
$$= x - v$$

C is called a cycle if $\partial(C) = 0$

Ex: $a + e - d$ is a cycle:

$$\partial(a+e-d) = \underbrace{\partial(a) + \partial(e)}_{x-v} - \partial(d) = (x-v) - (x-v) = 0$$

$a-b+c$ is also a cycle

C is called a boundary if $C=\partial(D)$ for some D

Ex: $\partial(\sigma) = d_0(\sigma) - d_1(\sigma) + d_2(\sigma)$
$= e - d + a = a+e-d$
so $a+e-d$ is a boundary

but $a-b+c$ is not a boundary.

Fundamental equation $\partial^2 = 0$

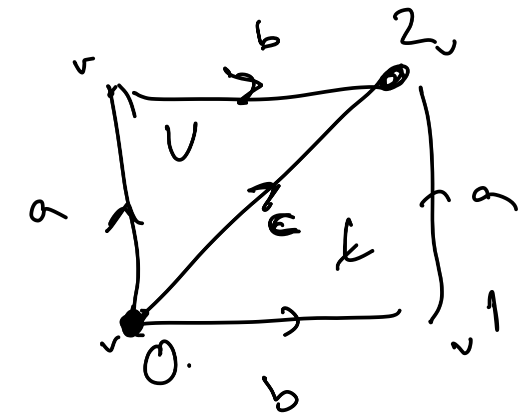
(i.e., $\partial_n \circ \partial_{n+1} = 0$), in other words every boundary is a cycle,

$$\underset{\substack{\uparrow \\ \text{boundaries} \\ \text{in } \Delta_{n+1}(K)}}{\operatorname{im} \partial_{n+1}} \subseteq \underset{\substack{\uparrow \\ \text{cycles} \\ \text{in } \Delta_n(K)}}{\ker \partial_n}$$

<u>Def</u>. The n^{th} homology group of K is the abelian group

$$H_n(K) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

Example



$K_0 = \{v\}$

$K_1 = \{a, b, c\}$

$K_2 = \{U, L\}$

$d_0(a) =$ endpoint

$d_1(a) =$ starting point

$d_0(U) = b$

$d_1(U) = c$

$d_2(U) = a$

$$\xrightarrow{\partial_3 = 0} \Delta_2(K) \xrightarrow{\partial_2} \Delta_1(K) \xrightarrow{\partial_1 = 0} \Delta_0(K) \xrightarrow{\partial_0} 0$$

$$\mathbb{Z}\{U, L\} \xrightarrow{\partial} \mathbb{Z}\{a, b, c\} \xrightarrow{\partial} \mathbb{Z}\{v\}$$

$$\left.\begin{aligned} \partial(U) &= a - c + b \\ \partial(L) &= b - c + a \end{aligned}\right\} \partial(U - L) = 0$$

$$\partial(a) = v - v = 0 = \partial(c) = \partial(b)$$

$$H_0(K) = \ker \partial_0 / \operatorname{im} \partial_1 = \mathbb{Z}\{v\}/_0 \cong \mathbb{Z}$$

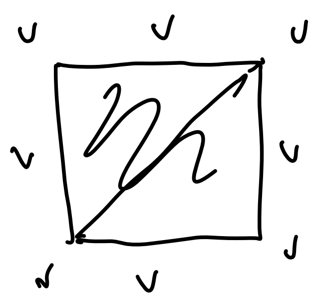
$$H_1(K) = \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z}\{a, b, c\}/_{(a - c + b)}$$

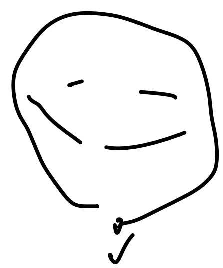
$c \equiv a + b$
$(\operatorname{mod} \operatorname{im} \partial_2)$

$$\cong \mathbb{Z}\{a, b\}$$
$$\cong \mathbb{Z}^2$$

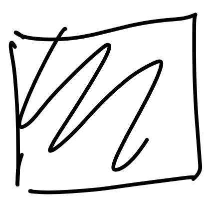
$$H_2(K) = \ker \partial_2 / \operatorname{im} \partial_3 = \ker \partial_2 = \mathbb{Z}(U - L)$$
$$\cong \mathbb{Z}$$

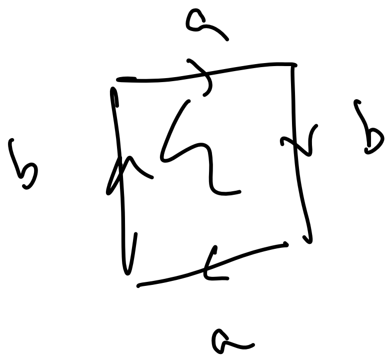
H_0	H_1	H_2
$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$





$\cong S^2$





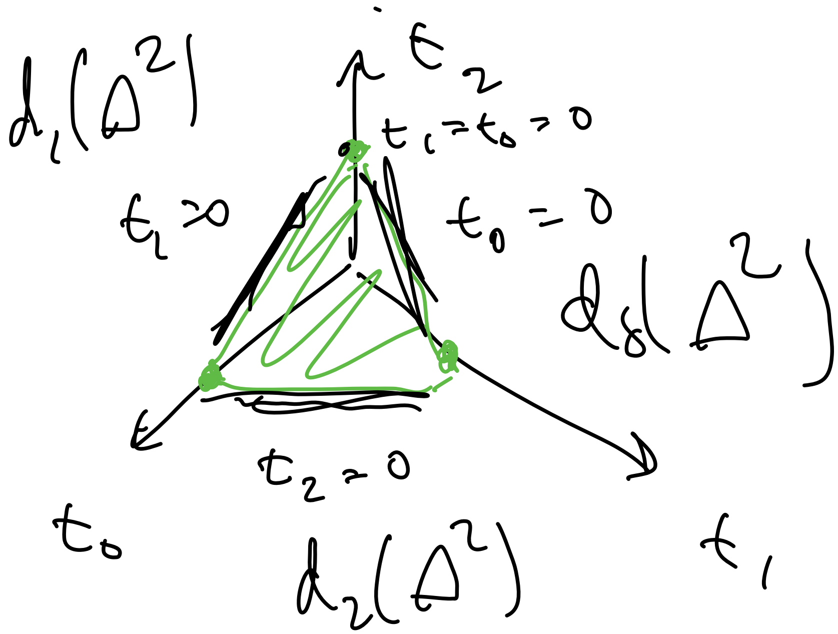
$\rightsquigarrow \mathbb{R}P^2$

$\rightsquigarrow$ Klein bottle

$$d^i : \Delta^{n-1} \longrightarrow \Delta^n$$

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} 0 \leq t_i \leq 1 \\ \sum t_i = 1 \end{array} \right\}$$

$\underline{n=2}$



$$d^1 : \Delta^1 \longrightarrow \Delta^2$$

$$(t_0, t_1) \mapsto (t_0, 0, t_1)$$

$$\boxed{\pi_1(X)^{ab} \cong H_1(X)} \qquad \pi_1(S^1) \cong \mathbb{Z}$$

$$\pi_1(T^2) \cong \mathbb{Z}^2$$

Hurewicz theorem