MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Optimering AN, 7,5 hp
January 12, 2022

You are allowed to bring an A-4 page (double sides) with whatever you think is important.
You must motivate well your arguments.

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1. (i) Show that the bounded closed interval $[a,b] = \text{conv}(\{a,b\})$.
   (ii) Let X be a nonempty bounded subset of $\mathbb{R}^n$. Define a function $S_X : \mathbb{R}^n \to \mathbb{R}$ by $S_X(x) = \sup\{\langle y, x\rangle : y \in X\}$. Show that $S_X(x)$ is a convex function.
   (iii) Show that $S_X = S_{\text{conv}(X)}$. Determine $S_{[a,b]}$ where $[a,b]$ is a closed interval on the real line.
   (iv) Show that $\langle \xi, d\rangle \le f'(x;d)$ for all $\xi \in \partial f(x)$ and all $d \in \mathbb{R}^n$. 13 p

2. Consider the following problem where $y = (y_1, ..., y_n)^t$, $y_0 = (\frac{1}{n}, ..., \frac{1}{n})^t$, and $e = (1, ..., 1)^t$ belong to $\mathbb{R}^n$: $\min\{y_1 : \|y - y_0\|^2 \le \frac{1}{n(n-1)}, e^t y = 1\}$. Write the KKT conditions for this problem and verify that $(0, \frac{1}{n-1}, ..., \frac{1}{n-1})^t$ is an optimal solution. Interpret this problem with respect to an inscribed sphere in the simplex defined by $\{y : e^t y = 1, y \ge 0\}$. 13 p

3. Consider the binary optimization problem: $\min\{x^T Q x : x_i \in \{-1, 1\}\}$, where $Q \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $x = (x_1, ..., x_n)^t$. Show that its Lagrange dual problem is in the following form: $\max\{\text{tr}\Lambda : Q - \Lambda \ge 0\}$, where Λ is a diagonal matrix.
   Hint: the constraint $x_i \in \{-1, 1\}$ is equivalent to $x_i^2 = 1$. 13 p

4. Use Phase I of the simplex method to determine whether the following system of equations has a nonnegative solution.
   $$4x_1 + 5x_2 + x_3 + 2x_4 = 0, 3x_1 + 3x_2 + x_3 + x_4 = 1$$
   Find one such solution, if it exists. 13 p

5. Formulate the minimization problem Minimize $\|Ax - b\|_\infty$ (ℓ_∞-norm approximation) as an LP problem. Explain in detail the relation between the optimal solution and the solution of its equivalent LP. 12 p

You have finished the exam if your homework $p_h \ge 24$. Continue otherwise

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at x and let the vectors $d_1, ..., d_n$ in $\mathbb{R}^n$ be linearly independent. Assume that the minimum of $f(x + \lambda d_j)$ over $\lambda \in \mathbb{R}$ occurs at $\lambda = 0$ for $j = 1, ..., n$. Show that $\nabla f(x) = 0$. Does this imply that f has a local minimum at x? 12 p

You have finished the exam if your homework $23 \ge p_h \ge 16$. Continue otherwise.

7. (i) Show, using the theory developed in this course, $G(x) \le A(x)$.
   (ii) Justify if the set $\{x \in \mathbb{R}^n_{++} : G(x) \ge A(x)\}$ is convex or not. Is this set a cone if we define $0^{\frac{1}{n}} = 0$?
   12 p

You have finished the exam if your homework $15 \ge p_h \ge 8$. Continue otherwise.

8. In line search to find optimal solution to nonlinear optimization problems we often need to solve the the following optimization problem
   $$\min\{\| -\nabla f(x) - d\|^2 : A_1 d = 0\},$$
   where A_1 is a $\nu \times n$ matrix with rank ν and x is fixed.
   (i) Find the optimal solution $\bar{d}$ without using the KKT conditions or Lagrange relaxation.
   (ii) Solve $\bar{d}$ from the KKT system.
   (iii) Find $\bar{d}$ in case $\nabla f(x) = (2, -3, 3)^t$ and $A_1 = \begin{pmatrix} 2 & 2 & -3 \\ 2 & 1 & 2 \end{pmatrix}$.
   12 p

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