STOCKHOLM UNIVERSITY
Department of Mathematics
Examiner: Salvador Rodríguez-López
MM5021-Foundations of Analysis

Final Written Exam
13 January, 2022
09:00-14:00

Please READ CAREFULLY the general instructions:

- During the exam you CAN NOT use any textbook, class notes, or any other supporting material.
- Calculators are **not allowed** during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. JUSTIFY your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).

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1. (a) **[1pt]** Define the notion of *complete* metric space.

   (b) **[2pt]** Show that if (X,d) is a complete metric space, and $E \subset X$ is a closed set, then E is complete as a metric space with the metric induced by d.

   (c) **[2pt]** Suppose that (X,d_X) and (Y,d_Y) are two metric spaces such that (X,d_X) is complete. Prove that if E is a closed set in X and $f : E \to Y$ is continuous such that

   $$d_Y(f(x), f(y)) \geq d_X(x,y),$$

   for all $x,y \in E$, then $f(E)$ is closed in Y.

2. (a) **[1pt]** Define what it means to say that a function is Riemann-Stieltjes integrable over an interval $[a,b]$ with respect to a monotonically increasing function α.

   (b) **[2pt]** Let $f : [a,b] \to \mathbb{R}$ be a monotonic function, and α an increasing continuous function defined on the same interval. Prove that $f \in \mathscr{R}(\alpha)$ over the interval $[a,b]$.

   (c) **[2pt]** Argue why the following limit exists, and calculate its value

   $$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \sin\left(\frac{k\pi}{2n}\right)}{n}.$$

3. (a) **[1pt]** Define what it means that a sequence of complex-valued functions defined on a metric space X converges uniformly on X to a function.

   (b) **[2pt]** For all $n \geq 1$ define $f_n(x) = (n+1)x^n(1-x)$ for $x \in [0,1]$. Show that the sequence converges pointwise to 0. Does it converge uniformly?

   (c) **[2pt]** Consider the series of functions given by

   $$\sum_{n\geq 1} \frac{(-1)^n}{n^3} \sin(2\pi n x).$$

   Show that it is continuous and differentiable on $\mathbb{R}$, and such that f' is also continuous on $\mathbb{R}$. Argument fully your answer.

4. Determine which of the following statements are true, and which are false. Explain your reasoning, by giving a proof or a counterexample to each statement. Each answer is graded over one point.

   i. If a set is not uncountable, then it is countable.

   ii. Let $(a_k)_k$ be a sequence of complex numbers. If the power series given by $\sum_{k\geq 0} a_k z^k$, converges for $z = 2$, it also converges for $z = 1 - i$.

   iii. If $\{V_\alpha\}_{\alpha\in I}$ is an finite family of open sets in a metric space, then $\cap_\alpha V_\alpha$ is open.

   iv. Given $f(x) = 1/(1+x^2)$ defined for $x \in \mathbb{R}$, it is satisfied that for every compact set $K \subset \mathbb{R}$, $f^{-1}(K)$ is a compact set.

   v. The function $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0 & ,\text{if } (x,y) = (0,0) \end{cases}$ is differentiable on $\mathbb{R}^2$.

1) a) See the course book

b) See the course book

c) We want to see that $f(E) \overset{?}{=} \overline{f(E)}$

and we know that $\overline{f(E)} = f(E) \cup f(E)'$.

So it suffices to show that for all

$y \in f(E)'$ then $\exists x \in E : f(x) = y$.

If $y \in f(E)'$, there exists $(x_n)_n$; $x_n \in E$ $\forall n \geq 1$ such that $\lim_n f(x_n) = y$.

This is because, by definition, $\forall n \geq 1$ $\exists y_n \in f(E)$ (for which $\exists x_n \in E : f(x_n) = y_n$) with $y_n \neq y$: $d(y_n, y) < 1/n$.

In particular $(f(x_n))_n$ is Cauchy in Y.

By the assumption:

$$d(f(x), f(y)) \geq 2\, d(x,y),$$

it follows that $(x_n)_n$ is Cauchy in E.

Since E is complete, $\exists x \in E : \lim_n x_n = x$.

By continuity of f, $y = \lim_n f(x_n) = f(x)$.

Hence $y \in f(E)$.

2) a) See the coursebook

b) See the coursebook

c) Let $f : [0,1] \to \mathbb{R}$
$x \mapsto f(x) = \sin\left(\frac{\pi}{2}x\right)$

Note that for $x \in [0,1]$, the function f is strictly increasing. By (b), it's then Riemann integrable on $[0,1]$. This yields in particular that if $P = \{\frac{k}{n}\}_{k=0,\dots,n}$ is a partition of $[0,1]$. we have that.

$$L(P_n, f) = \sum_{k=1}^{n} f\left(\frac{k-1}{n}\right) \cdot \frac{1}{n} = \sum_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right) \cdot \frac{1}{n}.$$

$$U(P_n, f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \sum_{k=1}^{n} \sin(k\pi/n) \cdot \frac{1}{n}.$$

and

$$U(P_n, f) - L(P_n, f) = \frac{1}{n}$$

Hence

$$U(P_n, f) - \frac{1}{n} \leq \int_a^b f\, dx \leq U(P_n, f)$$

which yields

$$\left| \int_0^1 f\, dx - U(P_n, f) \right| \leq \frac{1}{n}.$$

It follows that

$$\lim_{n\to\infty} U(P_n, f) = \int_0^1 f(x)\, dx = \int_0^1 \sin\left(\frac{x\pi}{2}\right) dx$$

$$= \frac{-2}{\pi} \Big[\cos(x\pi/2)\Big]_0^1 = \frac{2}{\pi}$$

3) a) See the coursebook

b). Note that for all $n \geq 1$ $f_n(0) = f_n(1) = 0$.

For all $0 < x < 1$

$$|f_n(x)| = |x|^n (n+1)(1-x)$$
$$\leq (n+1)x^n,$$

which yields that $\lim_{n\to\infty} f_n(x) = 0$ $\forall x \in [0,1]$.

Note that

$$f_n'(x) = n(n+1)x^{n-1}(1-x) - (n+1)x^n$$
$$= (n+1)x^{n-1}(n - nx - x).$$

So f_n has a maximum at $x_n = \frac{n}{n+1}$.

In that case

$$f_n\left(\frac{n}{n+1}\right) = (n+1)\frac{n^n}{(n+1)^n}\left(1 - \frac{n}{n+1}\right)$$

$$= \frac{1}{(1+\frac{1}{n})^n}.$$

So

$$\|f_n\|_\infty = \frac{1}{(1+1/n)^n} \xrightarrow[n\to\infty]{} \frac{1}{e}.$$

Then, f_n does not converge uniformly to 0.

b) Define the formal series

$$f(x) = \sum_{n\geq 1} \frac{(-1)^n}{n^3} \sin(2\pi n x) \quad . \text{ Let } \underset{a,b\in\mathbb{R}}{a<b}.$$

Let $S_N(x) = \sum_{n=1}^{N} \frac{(-1)^n}{n^3} \sin(2\pi n x)$, for $N \geq 1$.

Note that S_N is continuous and differentiable in $\mathbb{R}$, for being a finite sum of such functions.

Moreover

$$S_N'(x) = 2\pi \sum_{n\geq 1}^{N} \frac{(-1)^n}{n^2} \cos(2\pi n x) \in \mathcal{C}[a,b].$$

Note that for all $n \geq 1$ and $x \in \mathbb{R}$

$$\left| \frac{(-1)^n \cos(2\pi n x)}{n^2} \right| + \left| \frac{(-1)^n}{n^3} \sin(2\pi n x) \right| \leq \frac{1}{n^2}$$

and the series $\sum_{n\geq 1} \frac{1}{n^2}$ converges.

By the M-test of Weierstrass

$$S_N' \rightrightarrows \sum \frac{(-1)^n}{n^2} \cos(2\pi n x) = g$$

&

$$S_N \rightrightarrows f \quad \text{as } N \to +\infty.$$

Hence, we have that f is differentiable in (a,b) and $f'(x) = \sum \frac{(-1)^n}{n^2} \cos(2\pi n x)$

for all $x \in (a,b)$, and $g \in \mathcal{C}[a,b]$.

Since a,b are arbitrary, we conclude that $f \in \mathcal{C}^1(\mathbb{R})$ and $f' \in \mathcal{C}(\mathbb{R})$

4) i) False

The set $E = \{1\} \subset \mathbb{R}$ is finite, and neither countable nor uncountable

ii) True (Seen similar in the course, so we omit the details here.

Hint: Use of the root criteria.

iii) True

Given $y \in \bigcap_\alpha V_\alpha$, for all α $y \in V_\alpha$.

Since V_α is open, $\exists r_\alpha > 0 : B(y, r_\alpha) \subset V_\alpha$.

Define $r = \min_\alpha r_\alpha > 0$ (there is a finite number of them). Then

$$B(y, r) \subset \bigcap_\alpha V_\alpha.$$

iv) False : $f^{-1}([0,1]) = \mathbb{R}$.

v) False

Note that

$$\lim_{h \to 0} f(h,h) = \lim_{h \to 0} \frac{h^2}{2h^2} = \frac{1}{2} \neq f(0,0)$$

So f is not continuous at $(0,0)$

This implies that f can't be differentiable at $(0,0)$, and then it's not on $\mathbb{R}^2$.