MT7041 – Markov chains and mixing times – exam

Date Tuesday December 21, 2021
Examiner Daniel Ahlberg
Tools None allowed.
Grading criteria The exam is divided into two parts, consisting of 20 and 40 points respectively. To pass the exam, a score of 14 or higher is required on Part I. If this is attained, then also Part II is corrected, and its score determines the grade. The grade corresponds to the scores on the different parts according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Each problem of Part I may give up to 5 points and every problem of Part II may give up to 10 points. Complete and well motivated solutions are required for full credit. Partial solutions may be awarded partial credit.

Part I

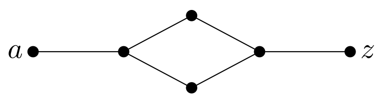
Problem 1. Let P be the transition matrix of a Markov chain on S. Show that for every $x, y \in S$ and $s, t \geq 1$ we have

$$P^{s+t}(x,y) \geq P^s(x,z)P^t(z,y) \quad \text{for every } z \in S.$$

Problem 2. Let $(X_t)_{t\geq 0}$ be a lazy simple random walk on $\mathbb{Z}$ and let n be a positive integer. Determine the limit

$$\lim_{t\to\infty} \mathbb{P}_0(X_t \text{ is a multiple of } n).$$

Problem 3. Consider a simple random walk on the graph below. Suppose that the walker starts at a. Determine the probability that the walker visits z before it returns to a.



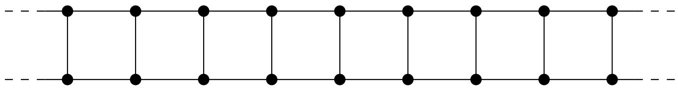
Problem 4. Consider a lazy simple random walk on the complete graph on five vertices, i.e. the graph with five vertices where every pair of vertices is joined by an edge. Compute $d(1)$ and $d(2)$, and determine the mixing time of the walker.

Part II

Problem 5. Let $(X_t)_{t\geq 0}$ be a Markov chain on a finite state space S. Show that given the 'present', 'tomorrow' is independent of 'yesterday'. That is, show that for every $A, B \subseteq S$, $x \in S$ and $t \geq 1$ we have

$$\mathbb{P}(X_{t+1} \in A, X_{t-1} \in B | X_t = x) = \mathbb{P}(X_{t+1} \in A | X_t = x)\mathbb{P}(X_{t-1} \in B | X_t = x).$$

Problem 6. Prove that the simple random walk on the infinite 'ladder graph' (depicted below) is recurrent.



Problem 7. Recall that a *shuffle* is any Markov chain on the symmetric group $(S_n, \circ)$ with transition matrix of the form $P(\sigma, \rho\sigma) = \mu(\rho)$ for $\rho \in S_n$, where μ is some probability measure on S_n. Let id denote the identity permutation, π the uniform distribution on S_n, and recall that

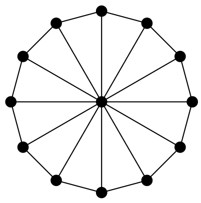
$$d(t) = \max_{\sigma \in S} \|P^t(\sigma, \cdot) - \pi\|_{TV}.$$

Prove that

(a) $P^t(\sigma, \rho\sigma) = P^t(id, \rho)$ for all $\rho, \sigma \in S_n$.

(b) $d(t) = \|P^t(id, \cdot) - \pi\|_{TV}$.

Problem 8. Consider a simple random walk on the graph obtained from a cycle of length n when connecting an additional vertex to all existing vertices.



(a) For $n \geq 3$, and any two vertices x and y, construct a coupling of two simple random walkers started at x and y such that they stay together once they first meet.

(b) Compute the expected coupling/coalescence time.

(c) Prove that there exists a constant C such that $t_{mix} \leq C$ for all $n \geq 3$.