Solutions exam MT7041 21 december, 2021

1 For every $x, y \in S$ and $s, t \geq 1$ we have

$$\begin{aligned}
P^{s+t}(x,y) &= \mathbb{P}(X_{s+t}=y \mid X_0=x) \\
\text{law of total probab} &= \sum_{v\in S} \mathbb{P}(X_{s+t}=y \mid X_s=v, X_0=x)\,\mathbb{P}(X_s=v, X_0=x) \\
\text{Markov prop} &= \sum_{v\in S} \mathbb{P}(X_{s+t}=y \mid X_s=v)\,\mathbb{P}(X_s=v \mid X_0=x) \\
&= \sum_{v\in S} P^t(v,y)\,P^s(x,v) \\
&\geq P^s(x,z)\,P^t(z,y) \qquad (5p)
\end{aligned}$$

for any specific $z \in S$.

2 Let $Y_t = X_t \pmod n$. Note that when $X_t = k + \ell n$ for some $\ell \in \mathbb{Z}$, then X_t jumps ± 1 with probab $1/4$ and remains with probab $1/2$. This corresponds to Y_t jumping ± 1 with probab $1/4$ each when $Y_t = k$, and otherwise remains. Hence $(Y_t)_{t\geq 0}$ is a SRW on $\mathbb{Z}_n$, which is the same as a SRW on a cycle of length n. (2p)

The chain is aperiodic since lazy, and irreducible since the graph connected. By symmetry, the uniform distribution is stationary. By the convergence theorem (2p)

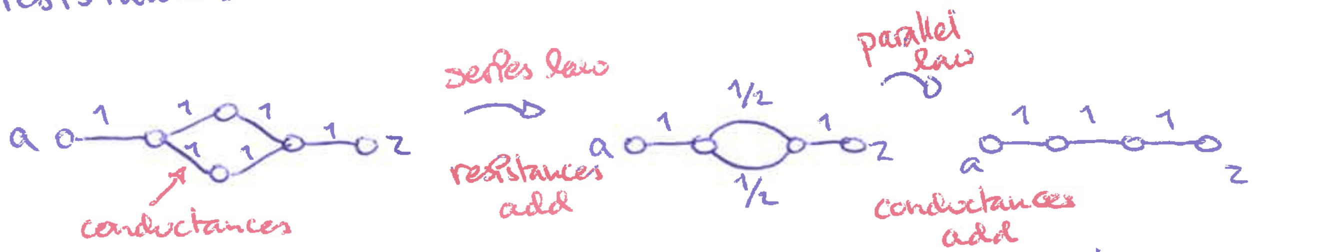
$$\lim_{t\to\infty} \mathbb{P}(X_t \text{ is a multiple of } n) = \lim_{t\to\infty} \mathbb{P}(Y_t = 0) = \underbrace{\frac{1}{n}}_{\text{uniform distr}}. \qquad (1p)$$

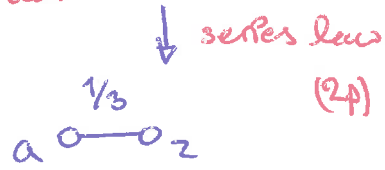
3 Consider the network obtained by assigning unit conductances to the edges of the graph. A theorem says that

$$\mathbb{P}_a(\tau_z < \tau_a^+) = \frac{1}{c(a)\cdot R(a \leftrightarrow z)} = \frac{1}{R(a\leftrightarrow z)}. \qquad (2p)$$

Using network reduction laws we may compute the effective resistance of the network.

The following operations do not change the effective resistance:





The effective conductance is 1/3 so the effective resistance is 3. We conclude that

$$\mathbb{P}_a(\tau_z < \tau_a^+) = 1/3. \qquad (1p)$$

Alt: Note that, by symmetry, the voltage at the center nodes equals 1/2, and since harmonic has to equal 1/3 and 2/3 at remaining nodes.

$\boxed{4}$ The graph has equal degree of every vertex, so a SRW has the uniform distribution as its stationary distribution.

Since all vertices are identical, we have

$$d(t) = \|P^t(x,\cdot) - \pi\|_{TV}.$$

$\pi(x) = 1/5$ for every $v \in S$. Also,

$$P(x,y) = \begin{cases} 1/2 & y = x \\ 1/8 & y \neq x \end{cases}$$

It follows that

$$d(1) \overset{\text{Prop}}{=} \frac{1}{2}\sum_{y\in S} |P(x,y) - \pi(y)| = \frac{3}{10} > \frac{1}{4}. \qquad (2p)$$

We next check $t=2$.

$$P^2(x,x) \overset{\text{law of total probab}}{=} \sum_{y\in S} P(x,y)P(y,x) = (P(x,x))^2 + 4\cdot P(x,y)P(y,x)$$

$$= \frac{1}{4} + 4\cdot\frac{1}{64} = \frac{5}{16}$$

$x \neq y$: $$P^2(x,y) = P(x,x)P(x,y) + P(x,y)P(y,y) + 3\cdot \underbrace{P(x,z)P(z,y)}_{z\neq x,y}$$

$$= \frac{1}{2}\cdot\frac{1}{8} + \frac{1}{8}\cdot\frac{1}{2} + 3\cdot\frac{1}{64} = \frac{11}{64}$$

Thus

$$d(2) = \frac{1}{2}\sum_{y\in S} |P(x,y) - \pi(y)| = \frac{9}{80} < 1/4. \qquad (2p)$$

Hence $t_{mix} = 2$. (1p)

$\boxed{5}$ Take $A, B \subseteq S$, $x \in S$ and $t \geq 1$.

$$\mathbb{P}(X_{t+1} \in A, X_{t-1} \in B \mid X_t = x)$$

additivity $= \sum_{y \in A} \sum_{z \in B} \sum_{x_0, \dots, x_{t-2} \in S} \mathbb{P}(X_{t+1} = y, X_{t-1} = z, X_{t-2} = x_{t-2}, \dots, X_0 = x_0 \mid X_t = x$ (2p)

condition $= \sum_{y \in A} \sum_{z \in B} \sum_{x_0, x_1, \dots, x_{t-2} \in S} \mathbb{P}(X_{t+1} = y \mid X_t = x, \dots, X_0 = x_0)\, \mathbb{P}(X_{t-1} = z, \dots, X_0 = x_0 \mid X_t =$ (2p)

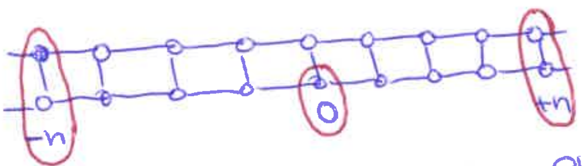
Markov property $= \sum_{y \in A} \mathbb{P}(X_{t+1} = y \mid X_t = x) \sum_{z \in B} \sum_{x_0, \dots, x_{t-2} \in S} \mathbb{P}(X_{t-1} = z, \dots, X_0 = x_0 \mid X_t = x)$ (4p)

additivity $= \mathbb{P}(X_{t+1} \in A \mid X_t = x)\, \mathbb{P}(X_{t-1} \in B \mid X_t = x)$. (2p)

$\boxed{6}$ We want to show that a SRW on the ladder graph is recurrent. Since all nodes are identical, it means we want to show that

$$\mathbb{P}_0(\tau_0^+ < \infty) = 1.$$

Let $\tau_{\pm n} := \min\{t \geq 0 : X_t \text{ reaches } \pm n\}$. Since the SRW

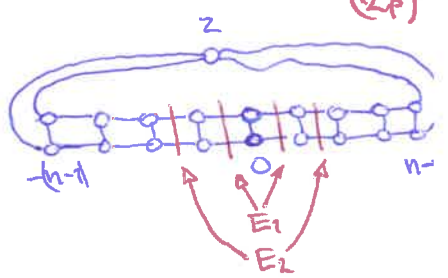


on a connected graph is irreducible it follows that $\mathbb{P}_0(\tau_{\pm n} < \infty) = 1$ for every $n \geq 1$. Hence

$$\mathbb{P}_0(\tau_0^+ = \infty) \leq \mathbb{P}_0(\tau_0^+ > \tau_{\pm n}) \quad \text{for all } n \geq 1. \quad (2p)$$

Suppose next that we replace the nodes at distance $\pm n$ with a node z connected to the nodes at distance $\pm(n-1)$, and nodes further from the origin are removed. Then the SRW on the ladder is identical to the SRW on the new graph until it first reaches $\pm n$, that is, $\tau_{\pm n}$ occurs. Hence (2p)

$$\mathbb{P}_0(\tau_0^+ > \tau_{\pm n}) = \mathbb{P}_0(\tau_0^+ > \tau_z)$$



According to a theorem, the probability on the RHS coincides with / can be expressed in terms of the effective resistance of the network obtained by assigning unit conductances to the edges of the graph. More precisely,

$$\mathbb{P}_0(\tau_0^+ > \tau_Z) = \frac{1}{c(0)\cdot R(0\leftrightarrow Z)} = \frac{1}{3\cdot R(0\leftrightarrow Z)}. \qquad (2p)$$

Using Nash-Williams inequality we may obtain a lower bound on the effective resistance. We identify n subsets $E_1, E_2, \ldots, E_n$ consisting of four edges each. Nash-Williams gives that

$$R(0\leftrightarrow Z) \geq \sum_{k=1}^{n} \Big(\sum_{e\in E_k} c(e)\Big)^{-1}$$
$$= n\cdot 4^{-1} = \frac{n}{4}. \qquad (2p)$$

Consequently

$$\mathbb{P}_0(\tau_0^+ = \infty) \leq \frac{1}{\frac{3}{4}\cdot n}.$$

Since this holds for all $n \geq 1$, the LHS equals zero. (2p)

$\boxed{7}$ (a) Note that for $t=1$ we have

$$P(\sigma, \rho\sigma) = \mu(\rho) = P(id, \rho) \qquad \forall \rho, \sigma \in S_n. \qquad (1p)$$

We proceed by induction. Suppose statement true for $t=k$. Then, for $t=k+1$, the law of total probability gives

$$P^{k+1}(\sigma, \rho\sigma) = \sum_{\sigma'\in S_n} P^k(\sigma,\sigma')P(\sigma',\rho\sigma) \qquad (1p)$$

Since there is a unique permutation $\bar{\sigma}$ that takes σ to σ' $(\bar{\sigma} = \sigma'\circ\sigma^{-1})$ we may rewrite the above expression as (1p)

$$= \sum_{\bar{\sigma}\in S_n} P^k(\sigma, \bar{\sigma}\sigma)P(\bar{\sigma}\sigma, \rho\sigma)$$

Induction hypothesis
$$= \sum_{\bar{\sigma}\in S_n} P^k(id, \bar{\sigma})\,\mu(\rho\bar{\sigma}^{-1}) \qquad (2p)$$
$$= \sum_{\bar{\sigma}\in S_n} P^k(id, \bar{\sigma})P(\bar{\sigma}, \rho)$$
$$= P^{k+1}(id, \rho).$$

Proof complete.

ⓑ By definition of the distance function $d(t)$ we have

$$d(t) = \max_{\sigma\in S_n} \| P^t(\sigma,\cdot) - \pi \|_{TV} \qquad (1p)$$

prop $\rightarrow$ $= \max_{\sigma\in S_n} \frac{1}{2}\sum_{\rho\in S_n} |P^t(\sigma,\rho) - \pi(\rho)|$ (2p)

equal since π is the uniform distribution

$\exists$ unique $\bar\sigma\in S_n$ that sends σ to ρ $\rightarrow$ $= \max_{\sigma\in S_n} \frac{1}{2}\sum_{\bar\sigma\in S_n} |P^t(\sigma,\bar\sigma\sigma) - \pi(\bar\sigma)|$

from part ⓐ $= \frac{1}{2}\sum_{\bar\sigma\in S_n} |P^t(id,\bar\sigma) - \pi(\bar\sigma)|$ (2p)

$$= \| P^t(id,\cdot) - \pi \|_{TV} \, .$$

8 ⓐ We suppose first that both x and y are both in the outer cycle and define a coupling as follows.

Let $X_0 = x$ and $Y_0 = y$. If $X_{t-1} \neq Y_{t-1}$, then let

- both chains jump to the center node with probab 1/3
- both chains jump clockwise / counter-clockwise with probab 1/3 each. (2p)

If $X_{t-1} = Y_{t-1}$, then as well make the same jump. Note that the chains meet when they first jump to the center node. Let τ_1 denote the number of steps needed for this to happen.

Suppose next that x is the center node and that y is in the outer cycle. Then move both chains independently until they are both in the outer cycle, and then repeat the coupling above. Let τ_2 denote the number of steps needed to both being in the outer cycle. (2p)

ⓑ The coupling time equals either τ_1 or $\tau_1+\tau_2$ depending on whether both the walkers start in the outer cycle or not. (To be precise, when one walker starts at the center the coupling time in fact equals $\tau_2 + \tau_1 \cdot \mathbb{1}\{X_{\tau_2} \neq Y_{\tau_2}\}$, but the above upper bound is fine.)

Note that $\tau_1 \sim geom(1/3)$ and $\tau_2 \sim geom(2/3)$, and have expectation 3 and 3/2 respectively. Hence (2p)

$\mathbb{E}[\tau_1] = 3$, $\mathbb{E}[\tau_1+\tau_2] = \frac{9}{2}$ (and $\mathbb{E}[\tau_2 + \tau_1 \mathbb{1}\{X_{\tau_2}\neq Y_{\tau_2}\}] = \frac{3}{2} + \frac{n-1}{n}\cdot 3$)

(c) Using the coupling method we have that

$$d(t) \leq \max_{x,y} \| P^t(x,\cdot) - P^t(y,\cdot) \|_{TV}$$

$$\leq \max_{x,y} \mathbb{P}(X_t \neq Y_t)$$

$$= \max_{x,y} \mathbb{P}(\tau_c > t) \qquad (2p)$$

Markov's inequality $\leq \max_{x,y} \frac{\mathbb{E}[\tau_c]}{t}$

from (b) $\leq \frac{9}{2 \cdot t} \qquad (2p)$

In particular we have $d(t) \leq 1/4$ when $t \geq 18$.

Thus $t_{mix} \leq 18$ for all $n \geq 1$.