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Solutions to Written Exam
Logic II
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Hints to solutions to written exam Logic II, 2019-02-27

1. For $y > 0$, use bounded search to find smallest $q(x, y)$ such that $x \dot{-} q(x, y)y < y$. Then $r(x, y) = x \dot{-} q(x, y)y$.

2. (a) Undecidable. Use Rice's theorem.
   (b) Undecidable. Use Rice's theorem.
   (c) Decidable. See simulation of Turing machines by partial recursive functions.

3. This is a contraposition to the well-known result that if a first-order theory has arbitrary large finite models, then it has only infinite models. See textbook.

4. The main forms of AC can be found in the textbook. First show that AC is equivalent to (AC'): for every surjective function $f \colon A \to B$, there is $g \colon B \to A$ such that $fg = \mathrm{id}_B$.

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5. (a) $|S_1| = 2^{2^{\aleph_0}}$
   (b) $|S_2| = 2^{\aleph_0}$ – see solution to corresponding problem 2019-01-16.
   (c) $S_3 \subseteq S_1$ which gives $|S_3| \leq |S_1| = 2^{2^{\aleph_0}}$. To show $2^{2^{\aleph_0}} \leq |S_3|$, it suffices to construct for each $M \subseteq (0, +\infty)$, a bijection $f_M : \mathbb{R} \to \mathbb{R}$ such that, if $f_M = f_{M'}$, then $M = M'$. This gives an injection $\mathcal{P}(0, +\infty) \to S_3$, which shows that $|\mathcal{P}(0, +\infty)| \leq |S_3|$. Now $\mathbb{R} \cong (0, +\infty)$ via the bijection $x \mapsto \exp(x)$, so $|\mathcal{P}(0, +\infty)| = |\mathcal{P}(\mathbb{R})| = 2^{2^{\aleph_0}}$. Such functions f_M can be constructed as

$$f_M(x) = \begin{cases} 0 & \text{if } x = 0 \\ x & \text{if } x \in M \text{ or } -x \in M \\ -x & \text{if } x \notin M \text{ or } -x \notin M \end{cases}$$

6. A suitable partial function is $f(x) \simeq \phi_x(x) + 1$. Then use the enumeration theorem to show that it cannot be extended to a total recursive function.

7. This equivalence can be found in the textbook.

8. The first part follows from Gödel's incompleteness theorem noticing that the axiom set $\mathrm{PA} \cup T$ is recursive, if T is finite. As for the second part, a counterexample is $T = \mathrm{Th}(\mathbb{N}) \supseteq \mathrm{PA}$, which is infinite, complete and consistent (but not recursive).