STOCKHOLMS UNIVERSITET
Matematiska institutionen
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Re-exam / Omtentamen
MM5024 Logik 7,5 hp
VT 2021
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This exam consists of two parts. The basic part (grundläggande del) has 7 problems (1–7), worth a total of 20 points. The problem part (problemdel) has 4 problems (8–11), worth a total of 20 points. You can obtain a maximum of 40 points. The questions are provided both in English (pp. 2–3) and Swedish (pp. 4–5).

You may submit your answers in either English or Swedish. Submissions should be uploaded as pdf assignments on the course website, either scanned or using LaTeX or similar. Further technical instructions can be found there.

The exam is open book: you can refer to the textbook, your own notes, and other reference resources. But you must not collaborate, discuss, or seek or receive assistance during the exam, either from each other within the course or from anyone else.

Write clearly and motivate your answers carefully. All answers should be fully justified (unless stated otherwise). You may use the soundness and completeness theorems (and any other theorems from the course), but state clearly when you do so.

——— Good luck! — Lycka till! ———

Written Exam (English)

Basic part

1 Which of the following formulas are tautologies? Justify your answers.

(a) $(P_1 \to P_2) \to (P_1 \vee P_2)$

(b) $(P_1 \wedge P_2) \to (P_1 \to P_2)$

2 Let $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ be the interpretations defined as follows:

	$P_1^{\mathcal{V}_i}$	$P_2^{\mathcal{V}_i}$
$\mathcal{V}_1$	1	1
$\mathcal{V}_2$	1	0
$\mathcal{V}_3$	0	0

(a) Give a formula which holds in $\mathcal{V}_1$ and $\mathcal{V}_2$, but not in $\mathcal{V}_3$

(b) Give a formula which holds in $\mathcal{V}_1$ and $\mathcal{V}_3$, but not in $\mathcal{V}_2$

3 Give the free variables of the following formulas:

(a) $(P_1(x_1) \to P_2(x_1)) \vee (P_2(x_1) \to P_1(x_1))$

(b) $\forall x_1\ (P_5(x_1) \to \exists x_2\ (P_5(x_2) \wedge P_{17}(x_2, x_1, x_2)))$

4 Give two formulas φ, ψ (over any arity type) such that $\varphi \approx \psi$, but $FV(\varphi) \neq FV(\psi)$.

5 (a) Find the error in the following derivation:

$$\dfrac{\dfrac{[\exists x_1 P_1(x_1, x_0)]^1 \qquad \dfrac{[P_1(x_1, x_0)]^2}{\exists x_0 P_1(x_0, x_0)}\ \exists I}{\exists x_0 P_1(x_0, x_0)}\ \exists E_2}{\exists x_1 P_1(x_1, x_0) \to \exists x_0 P_1(x_0, x_0)}\ \to I_1$$

(b) Is there any derivation of $\exists x_1 P_1(x_1, x_0) \to \exists x_0 P_1(x_0, x_0)$ without undischarged assumptions? Justify your answer.

6 Give a derivation showing that $(\exists x_1\ (P_1(x_1) \vee P_2(x_1))) \to ((\exists x_1\ P_1(x_1)) \vee (\exists x_1\ P_2(x_1)))$.

7 Consider the structure $\mathcal{Z} := \langle \mathbb{Z}\,;\ \leq\,;\ \ \rangle$, for the arity type $\langle 2\,;\ \ \rangle$, and let v_1, v_2 be the valuations of variables given by $v_1(x_i) = 2i$, $v_2(x_i) = 3i$, for all $i \in \mathbb{N}$.

Give a formula φ that holds in $\mathcal{Z}$ with the valuation v_2, but not with v_1

Problem part

8 Here is a possible alternative form of the elimination rule for $\wedge$:

$$\dfrac{\varphi \wedge \psi \qquad \begin{matrix}[\varphi],\ [\psi] \\ \vdots \\ \sigma\end{matrix}}{\sigma}\ \wedge E\text{-ALT}$$

in which φ, ψ, σ may be any formulas, and the rule may discharge any assumptions of φ or ψ used in the derivation of the premise σ.

If we had included this rule in the rules of natural deduction, we would require an extra case for it in the inductive proof of the soundness theorem. Give that case.

9 Given formulas φ, ψ, σ, consider the theory

$$\Gamma := \{\ \neg(\varphi \leftrightarrow \psi)\ ,\ \neg(\psi \leftrightarrow \sigma)\ ,\ \neg(\varphi \leftrightarrow \sigma)\ \}.$$

(a) Give a derivation showing that Γ is inconsistent.

(b) Give an alternative argument showing that Γ is inconsistent, without explicitly giving a derivation.

10 Which of the following formulas are derivable without undischarged assumptions? For each, give a derivation or a countermodel.

(a) $\forall x_1, x_2, x_3\ (x_1 \doteq x_2 \to f_1(x_1, x_3) \doteq f_1(x_2, x_3))$

(b) $(\forall x_1, x_2\ f_1(x_1, x_2) \doteq f_1(x_2, x_1)) \to (\forall x_1, x_2, x_3\ f_1(x_1, f_1(x_2, x_3)) \doteq f_1(f_1(x_3, x_2), x_1))$

(c) $(\forall x_1\, \exists x_2\, f_1(x_2) \doteq x_1) \to (\exists x_1\, \forall x_2\, f_1(x_2) \doteq x_1)$

11 (a) If $\Gamma \cup \{\varphi_1\}$ and $\Gamma \cup \{\varphi_2\}$ are both consistent theories, does it follow that $\Gamma \cup \{\varphi_1, \varphi_2\}$ must be consistent?

(b) If $\Gamma \cup \{\varphi_1\}$ and $\Gamma \cup \{\varphi_2\}$ are both *maximally* consistent theories, does it follow that $\Gamma \cup \{\varphi_1, \varphi_2\}$ must be maximally consistent?

——— End of exam ———

Skriftligt prov (Svenska)

Grundläggande del

1 Vilken av följande formler är tautologier? Motivera svaren.

(a) $(P_1 \to P_2) \to (P_1 \vee P_2)$

(b) $(P_1 \wedge P_2) \to (P_1 \to P_2)$

2 Låt $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ vara tolkningarna som definieras enligt följande:

	$P_1^{\mathcal{V}_i}$	$P_2^{\mathcal{V}_i}$
$\mathcal{V}_1$	1	1
$\mathcal{V}_2$	1	0
$\mathcal{V}_3$	0	0

(a) Ge en formel som gäller i $\mathcal{V}_1$ och $\mathcal{V}_2$, men inte i $\mathcal{V}_3$

(b) Ge en formel som gäller i $\mathcal{V}_1$ och $\mathcal{V}_3$, men inte i $\mathcal{V}_2$

3 Ange de fria variablerna i följande formler:

(a) $(P_1(x_1) \to P_2(x_1)) \vee (P_2(x_1) \to P_1(x_1))$

(b) $\forall x_1\ (P_5(x_1) \to \exists x_2\,(P_5(x_2) \wedge P_{17}(x_2, x_1, x_2)))$

4 Ange två formler φ, ψ (över vilken ställighetstypen som helst) så att $\varphi \approx \psi$, men $FV(\varphi) \neq FV(\psi)$.

5 (a) Hitta felet i följande härledning:

$$\dfrac{\dfrac{[\exists x_1 P_1(x_1, x_0)]^1 \qquad \dfrac{[P_1(x_1, x_0)]^2}{\exists x_0 P_1(x_0, x_0)}\ \exists I}{\exists x_0 P_1(x_0, x_0)}\ \exists E_2}{\exists x_1 P_1(x_1, x_0) \to \exists x_0 P_1(x_0, x_0)}\ \to I_1$$

(b) Finns det någon härledning av $\exists x_1 P_1(x_1, x_0) \to \exists x_0 P_1(x_0, x_0)$, utan oavslutade antagenden? Motivera svaret.

6 Ange en härledning som visar att $(\exists x_1\,(P_1(x_1) \vee P_2(x_1))) \to ((\exists x_1\, P_1(x_1)) \vee (\exists x_1\, P_2(x_1)))$.

7 Betrakta strukturn $\mathcal{Z} \coloneqq \langle \mathbb{Z}\,;\, \leq\,;\quad \rangle$, för ställighetstypen $\langle\, 2\,;\quad \rangle$, och låt v_1, v_2 vara värderingarna av variabler definierade av $v_1(x_i) = 2i$, $v_2(x_i) = 3i$, för alla $i \in \mathbb{N}$.

Ange en formel φ som gäller i $\mathcal{Z}$ med värderingen v_2, men inte med v_1

Problemdel

8 Här är en möjlig alternativ form av eliminationsregeln för $\wedge$:

$$\dfrac{\varphi \wedge \psi \qquad \begin{array}{c}[\varphi],\ [\psi] \\ \vdots \\ \sigma\end{array}}{\sigma}\ \wedge E\text{-ALT}$$

i vilken φ, ψ, σ kan vara godyckliga formler, och regeln kan avsluta antaganden av φ eller ψ som används i härledningen av premissen σ.

Om den här regeln hade inkluderats i reglerna för naturlig deduktion, så skulle vi behöva ett till fall i det induktiva beviset av sundhetssatsen. Ange det fallet.

9 För godtyckliga formler φ, ψ, σ, betrakta teorin

$$\Gamma := \{\ \neg(\varphi \leftrightarrow \psi)\ ,\ \neg(\psi \leftrightarrow \sigma)\ ,\ \neg(\varphi \leftrightarrow \sigma)\ \}.$$

(a) Ange en härledning som visar att Γ är inkonsistent.

(b) Ange en alternativ resonemang som visar att Γ är inkonsistent, utan att ange explicit en härledning.

10 Vilka av följande formler kan härledas utan oavslutade antagenden? För var och en, ange en härledning eller en motmodell.

(a) $\forall x_1, x_2, x_3\ \ (x_1 \doteq x_2 \to f_1(x_1, x_3) \doteq f_1(x_2, x_3))$

(b) $(\forall x_1, x_2\ f_1(x_1, x_2) \doteq f_1(x_2, x_1)) \to (\forall x_1, x_2, x_3\ f_1(x_1, f_1(x_2, x_3)) \doteq f_1(f_1(x_3, x_2), x_1))$

(c) $(\forall x_1\, \exists x_2\, f_1(x_2) \doteq x_1) \to (\exists x_1\, \forall x_2\, f_1(x_2) \doteq x_1)$

11 (a) Om teorierna $\Gamma \cup \{\varphi_1\}$ är $\Gamma \cup \{\varphi_2\}$ är både konsistenta, följer det att $\Gamma \cup \{\varphi_1, \varphi_2\}$ måste vara konsistent?

(b) Om teorierna $\Gamma \cup \{\varphi_1\}$ är $\Gamma \cup \{\varphi_2\}$ är både *maximalt* konsistenta, följer det att $\Gamma \cup \{\varphi_1, \varphi_2\}$ måste vara maximalt konsistent?

——— Slut på provet ———