Singular simplices

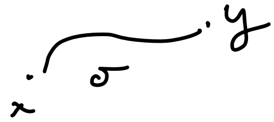
X topological space.

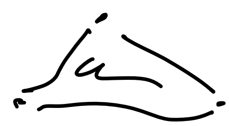
Def. A singular n-simplex in X is a continuous map $\sigma : \Delta^n \to X$.

Let $S_n(X)$ denote the set of all singular n-simplices in X.

$S_0(X)$ = set of points in X

$S_1(X)$ = set of paths in X



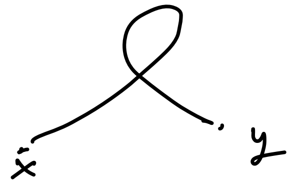
$S_2(X)$ = set of shapes like  in X

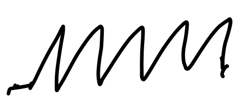
$\vdots$

Remark: Continuous maps can be wild!

For instance a singular 1-simplex (i.e. path) in X can be

- self-intersecting



- non-smooth 

- degenerate .
- space-filling 

Also, $S_n(X)$ is in general huge, even if X is "nice".

The singular Δ-complex $S(X)$

For $n \geq 1$ we have functions

$$d_i : S_n(X) \longrightarrow S_{n-1}(X) \qquad 0 \leq i \leq n$$

defined by:

$$\sigma : \Delta^n \to X \quad \in \quad S_n(X)$$

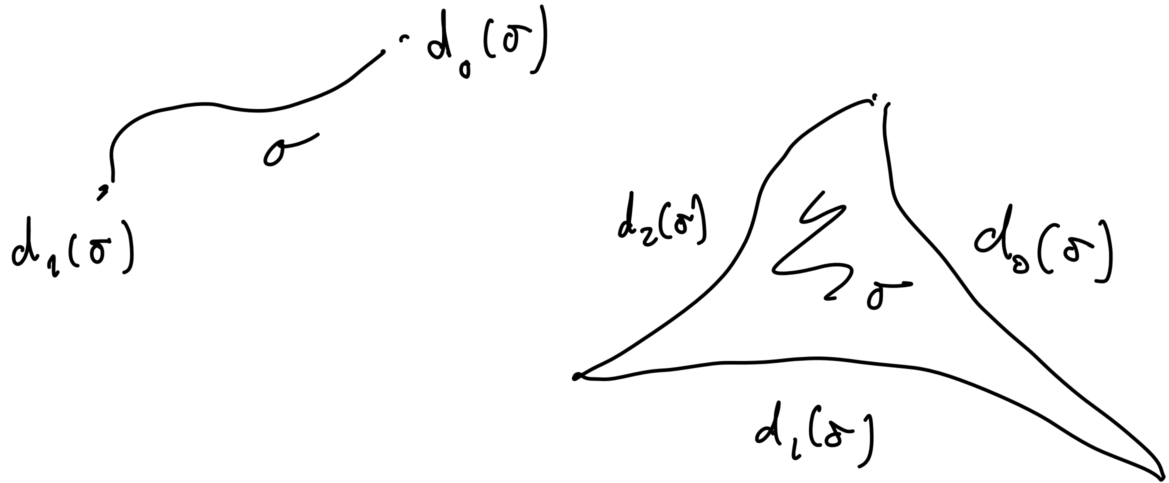
$$d_i(\sigma) : \Delta^{n-1} \to X \quad \in \quad S_{n-1}(X)$$

$$d_i(\sigma) = \sigma \circ d^i, \quad \text{where}$$

$$d^i : \Delta^{n-1} \longrightarrow \Delta^n \text{ is } d^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

$$d_i(\sigma)(t_0, \dots, t_{n-1}) = \sigma(t_0, \dots, 0, \dots t_{n-1}).$$

Ex.



Exercise: $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$.
(as functions $\Delta^{n-2} \to \Delta^n$).

Thus, $S_0(X)$, $S_1(X)$, $S_2(X)$, ...
together with the maps d_i
form an (abstract) Δ-complex!

The singular chain complex $C_*(X)$

$$C_n(X) = \mathbb{Z}\, S_n(X) \quad (= \Delta_n(S(X)))$$

free abelian group on $S_n(X)$

Elements of $C_n(X)$ are finite linear combinations $\sum_{\sigma \in S_n(X)} n_\sigma \cdot \sigma$, $n_\sigma \in \mathbb{Z}$,

with $n_\sigma = 0$ for all but finitely many $\sigma \in S_n(X)$.

Boundary homomorphism

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i d_i(\sigma)$$

Def. The singular homology groups of X are defined by

$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

<u>Functoriality</u> $f: X \to Y$ continuous.

there is an induced function

$$f_\#: S_n(X) \longrightarrow S_n(Y), \quad f_\#(\sigma) = f \circ \sigma$$

$$\Delta^n \xrightarrow{\sigma} X \quad \longmapsto \quad \begin{tikzcd} \Delta^n \arrow[rr, "f_\#(\sigma)"] \arrow[dr, "\sigma"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

and hence an induced homomorphism

$$f_\#: C_n(X) \to C_n(Y)$$

$$f_\#(n_1\sigma_1 + \dots + n_k\sigma_k) = n_1 f_\#(\sigma_1) + \dots + n_k f_\#(\sigma_k)$$

Moreover, the diagram

$$\begin{tikzcd} S_n(X) \arrow[r, "f_\#"] \arrow[d, "d_i"'] & S_n(Y) \arrow[d, "d_i"] \\ S_{n-1}(X) \arrow[r, "f_\#"] & S_{n-1}(Y) \end{tikzcd}$$

commutes, i.e., $f_\# \circ d_i = d_i \circ f_\#$.

This implies that

$$\begin{array}{ccc} C_n(X) & \xrightarrow{f_\#} & C_n(Y) \\ \partial_n \downarrow & & \downarrow \partial_n \\ C_{n-1}(X) & \xrightarrow{f_\#} & C_{n-1}(Y) \end{array}$$

commutes as well.

This means that $f_\# : C_*(X) \to C_*(Y)$ is a <u>chain map</u>, i.e.,

$$f_\# \circ \partial = \partial \circ f_\# .$$

This means in particular:

- $c \in C_n(X)$ cycle, $\partial(c) = 0 \Rightarrow f_\#(c) \in C_n(Y)$ cycle

  (because $\partial(f_\#(c)) = f_\#(\underbrace{\partial(c)}_{=0}) = 0$)

- $c \in C_n(X)$ boundary, i.e., $c = \partial(D)$ for some $D \in C_{n+1}(X)$

  $\Rightarrow f_\#(c) \in C_n(Y)$ boundary.

  (because $f_\#(c) = f_\#(\partial(D)) = \partial(f_\#(D))$).

This implies that

$$f_*: H_n(X) \to H_n(Y)$$

defined by

$$f_*[C] = [f_\#(C)]$$

is a well-defined homomorphism.

(Here $C \in \ker \partial_n$, $[C] = C + \operatorname{im} \partial_{n+1}$
$\in \underbrace{\ker \partial_n / \operatorname{im} \partial_{n+1}}_{= H_n(X)}$)

$[C] = [C']$ means $C - C' = \partial D$ for some D. Then

$$f_\#(C) - f_\#(C') = \partial f_\#(D)$$

$\Rightarrow$ $[f_\#(C)] = [f_\#(C')]$.

Moreover, H_n is a <u>functor</u> from the category of topological spaces and continuous maps to the category of abelian groups and homomorphisms:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$(g \circ f)_* = g_* \circ f_* : H_n(X) \to H_n(Z),$$ and

$1_X : X \to X$ identity map

$$(1_X)_* = 1_{H_n(X)} : H_n(X) \to H_n(X).$$

<u>Consequence:</u> $H_n(X)$ is a topological invariant!
If $X \cong Y$ (homeomorphic)
then $H_n(X) \cong H_n(Y)$ (isomorphic).

<u>Reason:</u> $X \cong Y$ means that there are cont. maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that

$f \circ g = 1_Y$, and $g \circ f = 1_X$

This implies

$$H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(X)$$

$$f_* \circ g_* = (f \circ g)_* = (1_Y)_* = 1_{H_n(Y)}$$

similarly, $g_* \circ f_* = 1_{H_n(X)}$.

This means f_* is an isomorphism (with inverse g_*).

<u>Remark</u>: The converse is not true

There are examples of maps

$$f : X \to Y$$

such that $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all n, but f is <u>not</u> a homeomorphism.

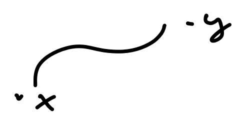
<u>Remark</u>: K Δ-complex. Then we will show later that

$$H_n(|K|) \cong H_n(K)$$

$$\left(\overset{\shortparallel}{H_n^{\Delta}(|K|)} \right)$$

$H_0(X)$

<u>Reminder</u>: A path in X is a cont. map
$\sigma : \Delta^1 \to X$.
The path-component of $x \in X$ is the subspace
$$P_x = \{ y \in X \mid \exists \text{ path } \sigma \text{ in } X \text{ with } d_1(\sigma) = x,\ d_0(\sigma) = y \}$$



$$\pi_0(X) := \text{set of path components}$$
$$= \{ P_x \mid x \in X \}$$

$$X = \bigcup_{P_x \in \pi_0(X)} P_x \qquad \text{disjoint union (as a set)}$$

X is path-connected if $X = P_x$ for any $x \in X$
$\Leftrightarrow \pi_0(X) = \{X\}$.

Define a function $\pi_0(X) \longrightarrow H_0(X)$
$$P_x \longmapsto [x].$$
This is well-defined because:
$P_x = P_y \Leftrightarrow \exists\, \sigma : \Delta^1 \to X \quad d_0(\sigma) = x,\ d_1(\sigma) = y$
$\Rightarrow \quad \partial_1(\sigma) = d_0(\sigma) - d_1(\sigma) = x - y$

$\Rightarrow \quad [x] = [y] \in H_0(X)$.

This induces a homomorphism

$$\underbrace{\mathbb{Z}\pi_0(X)}_{\substack{\text{free abelian} \\ \text{group on } \pi_0(X)}} \to H_0(X)$$

<u>Theorem</u>: This is an isomorphism.

<u>Proof</u>: First step: show that

$$H_0(X) \cong \bigoplus_{P_x \in \pi_0(X)} H_0(P_x) \quad \text{(see Hatcher prop. 2.6.)}$$

similarly,

$$\pi_0(X) = \coprod_{P_x \in \pi_0(X)} \{P_x\} \quad \Rightarrow$$

$$\mathbb{Z}\pi_0(X) \cong \bigoplus_{P_x \in \pi_0(X)} \mathbb{Z}\{P_x\}$$

$\rightsquigarrow$ we may assume X path-conn.

second step:

Proposition Assume X path-connected. Then $H_0(X) \cong \mathbb{Z}$ (spanned by $[x]$ for any $x \in X$). $X \neq \emptyset$

Proof: Define $\varepsilon : C_0(X) \to \mathbb{Z}$ by

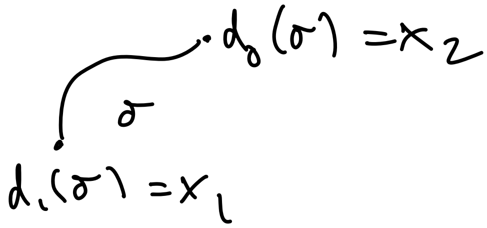
$$\varepsilon(n_1 x_1 + \dots + n_k x_k) = n_1 + \dots + n_k .$$

for points $x_1, \dots x_n \in X$ (i.e. 0-simplices)

ε surjective: Pick $x \in X$ $\quad \varepsilon(n \cdot x) = n$
$n \in \mathbb{Z}$.

Claim: $\ker(\varepsilon) = \operatorname{im}(\partial_1 : C_1(X) \to C_0(X))$.

$\supseteq$: $\varepsilon \partial_1(\sigma) = \varepsilon(d_0(\sigma) - d_1(\sigma))$

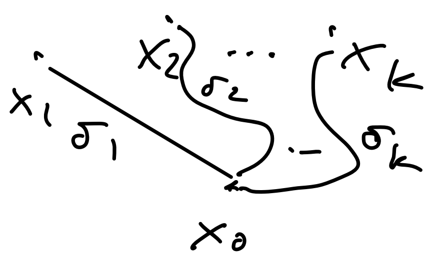


$$= \varepsilon(1 \cdot x_2 - 1 \cdot x_1)$$
$$= 1 - 1 = 0.$$

$\subseteq$: Suppose $\varepsilon(n_1x_1 + \cdots + n_kx_k) = 0$, i.e.,
$n_1 + \cdots + n_k = 0$.

Let $x_0 \in X$ be any point.

Since X path-connected, we can find paths $\sigma_i : \Delta^1 \to X$ with $(1 \le i \le k)$
$d_0(\sigma_i) = x_i$ and $d_1(\sigma_i) = x_0$



Let $c = n_1\sigma_1 + \cdots + n_k\sigma_k$
$\in C_1(X)$

Then

$$\partial(c) = n_1\partial(\sigma_1) + \cdots + n_k\partial(\sigma_k)$$

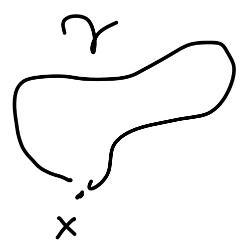
$$= n_1(x_1 - x_0) + \cdots + n_k(x_k - x_0)$$

$$= n_1x_1 + \cdots + n_kx_k - \underbrace{(n_1 + \cdots + n_k)}_{=0}\cdot x_0$$

Hence, $n_1x_1 + \cdots + n_kx_k \in \operatorname{im} \underline{\partial_1}$

$H_1(X)$ (Cursory; see Hatcher §2A)

A loop in X at $x \in X$ is the same thing as a singular 1-simplex $\gamma : \Delta^1 \to X$ with $d_0(\gamma) = d_1(\gamma) = x$.



This means that $\gamma \in C_1(X)$ is cycle:

$$\partial_1(\gamma) = d_0(\gamma) - d_1(\gamma) = x - x = 0.$$

so we can form $[\gamma] \in H_1(X)$.

<u>Recall</u>: $\pi_1(X, x) = \{\text{loops } \gamma \text{ at } x\}/\simeq$

homotopy

(Hurewicz homomorphism).

<u>Theorem</u>:
$$\pi_1(X, x) \longrightarrow H_1(X)$$
$$[\gamma]_{\simeq} \longmapsto [\gamma]$$

is a well-defined homomorphism of groups. Moreover, the kernel is the commutator subgroup of $\pi_1(X, x)$, so we get $\pi_1(X, x)^{ab} \xrightarrow{\cong} H_1(X)$.

Homotopy invariance of homology

Let $f, g : X \to Y$ be cont. maps.

A homotopy from f to g is a cont. map

$$H : X \times I \longrightarrow Y \qquad I = [0,1] \subseteq \mathbb{R}$$

such ~~that~~ $H(x,0) = f(x)$ $\forall x \in X$.
$H(x,1) = g(x)$

We say "f is homotopic to g" and write $f \simeq g$ if such an H exists.

Remark: • Set $f_t(x) = H(x,t)$.
Then $f_t : X \to Y$ is cont. for every $t \in [0,1]$
and $f_0 = f$, $f_1 = g$.

• $\simeq$ is an equivalence relation on the set of cont. maps $X \to Y$.

Homotopy equivalence

- $f : X \to Y$ is a homotopy equivalence if there is a map $g : Y \to X$ s.t. $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$.

- X and Y are called homotopy equivalent if there is a homotopy equivalence $f : X \to Y$. Write $X \simeq Y$ in this case.

Examples: • Every convex subspace $X \subseteq \mathbb{R}^n$ is contractible, i.e., $X \simeq *$ ← one-point space

In particular, $\mathbb{R}^n$, Δ^n are contractible.

- $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ ((n-1)-dim sphere)

Theorem If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$.

This will follow from:

Theorem: If $f \simeq g : X \to X'$, then $f_* = g_* : H_n(X) \to H_n(X')$ for all n.

Strategy: prove that $f \simeq g$ implies $f_\# \simeq g_\# : C_*(X) \to C_*(Y)$ chain homotopy.

The notes below were not part of the lecture but arose from the discussion after the lecture, so everything below is cursory.

$\mathbb{R}^n \simeq *$

$(\Rightarrow H_k(\mathbb{R}^n) \cong H_k(*) = \begin{cases} \mathbb{Z}, & k=0 \\ 0, & k>0 \end{cases})$

But $\mathbb{R}^n \not\cong *$

Not true in general that

$\forall n\ H_n(X) \cong H_n(Y) \Rightarrow X \simeq Y$.

But if X, Y are Δ-complex (or CW-complex)

and if X, Y are simply connected

(i.e., $\pi_1(X) = \pi_1(Y) = 0$)

and if $f: X \to Y$ induces isomorphism

$f_*: H_n(X) \to H_n(Y)$ for all n,

<u>Then</u> f is a homotopy equivalence.

(Whitehead theorem)

$\mathbb{CP}^2 \qquad H_k(\mathbb{CP}^2) \cong \begin{cases} \mathbb{Z}, & k = 0, 2, 4 \\ 0, & \text{otherwise} \end{cases}$

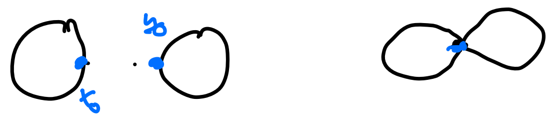
$S^2 \vee S^4 \qquad H_k(S^2 \vee S^4) \cong \begin{cases} \mathbb{Z}, & k = 0, 2, 4 \\ 0, & \text{otherwise} \end{cases}$

but $\mathbb{CP}^2 \not\simeq S^2 \vee S^4$.

$X \vee Y = X \sqcup Y /_{x_0 \sim y_0}$

$x_0 \in X$
$y_0 \in Y$

$S^1 \vee S^1$



$H_0(\mathbb{R} \times \mathbb{R}) = H_0(\mathbb{R}^2) \cong \mathbb{Z}$

$H_0(\mathbb{R}) \times H_0(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^2$