1. $\varphi := (P_1 \vee P_2) \leftrightarrow P_3$

$\chi := P_3 \to P_1$

Define V_1 by $P_1^{V_1} = 0$, $P_2^{V_1} = 1$, $P_3^{V_1} = 1$.

Then $[\![\varphi]\!]^{V_1} = 1$, $[\![\chi]\!]^{V_1} = 0$, as required.

Define V_2 by $P_1^{V_1} = 0$, $P_2^{V_1} = 0$, $P_3^{V_1} = 1$.

Then $[\![\varphi]\!]^{V_1} = 0$, $[\![\chi]\!]^{V_1} = 1$, as required.

2. (a) $P_1 \wedge P_2,\ P_1 \to P_3,\ P_2 \to P_4 \vdash P_3 \wedge P_4$:

$$\dfrac{\dfrac{P_1 \to P_3 \qquad \dfrac{P_1 \wedge P_2}{P_1}\,{\scriptstyle \wedge E}}{P_3}\,{\scriptstyle \to E} \qquad \dfrac{P_2 \to P_4 \qquad \dfrac{P_1 \wedge P_2}{P_2}\,{\scriptstyle \wedge E}}{P_4}\,{\scriptstyle \to E}}{P_3 \wedge P_4}\,{\scriptstyle \wedge I}$$

(b) $P_1 \leftrightarrow P_2,\ P_1 \to P_3,\ P_2 \to P_4 \nvdash P_3 \leftrightarrow P_4$;

this is <u>not</u> derivable (by soundness) since a counter-model is given by

$$P_1^V = P_2^V = P_3^V = 0, \quad P_4^V = 1.$$

3. (a) If φ, ψ are tautologies,
then in any interpretation, both hold,
so they certainly have the same truth-value.
So $\varphi \approx \psi$.

(b) "P_1" and "$\neg\neg P_1$" are logically equivalent
& not tautologies.

4. (a) $FV(\neg(P_1(x_1) \vee \neg P_1(x_2))) = \{x_1, x_2\}$

(b) $FV((\exists x_2, x_3\ P_2(x_1, x_2, x_3)) \to (\exists x_2\ P_1(x_1, x_2, x_2)))$
$= \{x_1\}$

5. (a)

$$\dfrac{\exists x_1\ P_1(x_1) \qquad \dfrac{[P_1(x_1)]}{\forall x_1\ P_1(x_1)}\,\forall I}{\forall x_1\ P_1(x_1)}\,\exists E$$

The error is in the $\forall I$ rule:
the general rule says "x_i may not occur free in any undischarged assumption"; here x_1 occurs free in the assumption $P_1(x_1)$, which is undischarged at

the point where $\forall I$ is used (though it is discharged later by $\exists E$).

(Another error is that if the derivation were valid, it would show $(\exists x_1\, P_1(x_1)) \vdash (\forall x_1\, P_1(x_1))$, not $\vdash (\exists x_1\, P_1(x_1)) \to (\forall x_1\, P_1(x_1))$ as the question suggests.)

(b) If $(\exists x_1\, P_1(x_1)) \to (\forall x_1\, (P_1(x_1)))$ were derivable without assumptions, it would be a tautology, by soundness. But a countermodel is given by the structure $\langle \mathbb{N};\, \underbrace{\text{even}};\, \rangle$.

(i.e. $[\![P_1(x)]\!] = \begin{cases} 1 & x \text{ is even} \\ 0 & \text{o/wise} \end{cases}$)

since there exist even naturals, but not all nat's are even.

6. (a) This may fail taking e.g. $s = x_1$, $t = x_2$, & the structure $\langle \mathbb{N};\, \text{even};\, \rangle$ as above, with $v(x_1) = 0$, $v(x_2) = 2$.

Then $[\![P_1(x_1) \leftrightarrow P_1(x_2)]\!] = 1$, but $[\![x_1 \doteq x_2]\!] = 0$.

(b) $s \doteq t \vdash P_1(s) \leftrightarrow P_1(t)$:

$$\dfrac{s \doteq t \qquad \dfrac{\dfrac{[P_1(s)]^1}{P_1(s) \to P_1(s)}\,{\to}I_1 \qquad \dfrac{[P_1(s)]^2}{P_1(s) \to P_1(s)}\,{\to}I_2}{P_1(s) \leftrightarrow P_1(s)}\,\wedge I}{P_1(s) \leftrightarrow P_1(t)}\,\text{repl}$$

7. Take δ to be "$\exists x_3 \ f_4(x_3, x_2) \doteq x_1$".

This will hold in $\mathcal{N}, v$ iff $\exists k \in \mathbb{N}$, $k\, v(x_2) = v(x_1)$,

i.e. iff $v(x_2) \mid v(x_1)$, as required.

8. (a) A typical instance of replacement is

$$\frac{s \doteq t \qquad \sigma[s/x_m]}{\sigma[t/x_m]}$$ with s, t free for x_m in ψ.

(using "ψ", "x_m" to avoid clashing "φ, x_i, x_j, x_k" in $=$-elim)

Using $=$-elim, we can derive this as:

$$\dfrac{\psi[s/x_m] \qquad \dfrac{s \doteq t \qquad \dfrac{[\psi]^1}{\psi \to \psi}\to I_1}{\psi[s/x_m] \to \psi[t/x_m]}\ {=}\text{-elim}}{\psi[t/x_m]}\to E$$

where for the instance of $=$-elim, we take:

- s & t are the same as in the given instance of repl;
- x_i & x_j are taken to be any variables not occurring in ψ;
- φ is taken to be $\psi[x_i/x_m] \to \psi[x_j/x_m]$
- x_k is taken to be x_m.

This ensures that the given freeness conditions are satisfied, & that

$$\varphi[x_k/x_i][x_k/x_j] = \chi \to \chi$$

$$\varphi[s/x_i][t/x_j] = \chi[s/x_m] \to \chi[t/x_m].$$

(b) We're inductively showing: For every derivation $\mathcal{D}$,

in any interp. A, v, if all ass'ns of $\mathcal{D}$ hold, then the conclusion of $\mathcal{D}$ holds. $\Big\}$ "$\mathcal{D}$ is sound"

Suppose $\mathcal{D}$ concludes with "=-elim":

$$\mathcal{D}\left\{\dfrac{\left.\begin{matrix}\vdots\\ s=t\end{matrix}\right\}\mathcal{D}_1 \qquad \left.\begin{matrix}\vdots\\ \varphi[x_k/x_i][x_k/x_j]\end{matrix}\right\}\mathcal{D}_2}{\varphi[s/x_i][t/x_j]}\right.$$

where x_k is not free in any ass'n of $\mathcal{D}_2$.

Assume as IH that $\mathcal{D}_1$, $\mathcal{D}_2$ are sound.

Let A, v be any interp'n in which all ass'ns of $\mathcal{D}$ hold; we must show $\varphi[s/x_i][t/x_j]$ holds there.

Since $=$-elim discharges no assumptions, all ass'ns of $\mathcal{D}_1$ hold in $\mathcal{A}, v$, so by the IH for $\mathcal{D}_1$,

$\mathcal{A}, v \models s = t$, i.e. $[\![s]\!]^v = [\![t]\!]^v$.

By the substitution lemma, & the given free-var. conditions,

$$[\![\varphi[s/x_i][t/x_j]]\!]^v = [\![\varphi]\!]^{v[x_i \mapsto [\![s]\!]^v][x_j \mapsto [\![t]\!]^v]}$$

$$= [\![\varphi]\!]^{v[x_i \mapsto [\![s]\!]^v][x_j \mapsto [\![s]\!]^v]}$$

(since $[\![s]\!]^v = [\![t]\!]^v$ above).

Since we are giving x_i, x_j the $\underline{\text{same}}$ value, we can see this as an evaluation of $\varphi[x_k/x_i][x_k/x_j]$.

Write ω for $v[x_k \mapsto [\![s]\!]^v]$.

Then $[\![x_k]\!]^\omega = [\![s]\!]^v$,

so $v[x_i \mapsto [\![s]\!]^v][x_j \mapsto [\![s]\!]^v]$ and $\omega[x_i \mapsto [\![x_k]\!]^\omega][x_j \mapsto [\![x_k]\!]^\omega]$ agree on all vars except x_k,

so (since x_k not free in φ),

$$[\![\varphi]\!]^{v[x_i \mapsto [\![s]\!]^v][x_j \mapsto [\![s]\!]^v]} = [\![\varphi]\!]^{\omega[x_i \mapsto [\![x_k]\!]^\omega][x_j \mapsto [\![x_k]\!]^\omega]}$$

$$= [\![\varphi[x_k/x_i][x_k/x_j]]\!]^\omega$$

So now by the IH for $\mathcal{D}_2$, it suffices to show all assumptions of $\mathcal{D}_2$ hold under w. We know they hold under v, since they are assumptions of $\mathcal{D}$; & since x_k is not free in them, and v and w agree on all vars except possibly x_k, they must also hold under v. So we are done.

9. $\neg(\forall x_1\ \varphi\wedge\psi)) \vdash \exists x_1\,(\neg\varphi\vee\neg\psi)$:

$$
\begin{prooftree}
\AxiomC{$\neg\forall x(\varphi\wedge\psi)$}
\AxiomC{$[\neg\exists x(\neg\varphi\vee\neg\psi)]^1$}
\AxiomC{$[\neg(\neg\varphi\vee\neg\psi)]^3$}
\AxiomC{$[\neg\varphi]^4$}
\RightLabel{$\vee I$}
\UnaryInfC{$\neg\varphi\vee\neg\psi$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{$\to I_4$}
\UnaryInfC{$\neg\neg\varphi$}
\AxiomC{$[\neg(\neg\varphi\vee\neg\psi)]^3$}
\AxiomC{$[\neg\psi]^6$}
\RightLabel{$\vee E$}
\UnaryInfC{$\neg\varphi\vee\neg\psi$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{$\to I_6$}
\UnaryInfC{$\neg\neg\psi$}
\AxiomC{$[\neg(\varphi\wedge\psi)]^2$}
\AxiomC{$[\varphi]^5$}
\AxiomC{$[\psi]^7$}
\BinaryInfC{$\varphi\wedge\psi$}
\BinaryInfC{$\bot$}
\RightLabel{$\to I_7$}
\UnaryInfC{$\neg\psi$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{$\to I_5$}
\UnaryInfC{$\neg\varphi$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{RAA_3}
\UnaryInfC{$\neg\varphi\vee\neg\psi$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x(\neg\varphi\vee\neg\psi)$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{RAA_2}
\UnaryInfC{$\varphi\wedge\psi$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x\ \varphi\wedge\psi$}
\RightLabel{$\to E$}
\BinaryInfC{$\bot$}
\RightLabel{RAA_1}
\UnaryInfC{$\exists x_1\ \neg\varphi\vee\neg\psi$}
\end{prooftree}
$$

10. Call the given structures $\mathcal{N}$, $\mathcal{R}$.

(a) Note that $\mathbb{N}$ has a least element, while $\mathbb{R}$ does not. So the formula

"$\exists x_1 \forall x_2 \; P_1(x_1, x_2)$"

holds in $\mathcal{N}$ but not in $\mathcal{R}$.

(b) φ_{pre} expresses the axioms of a pre-order: in a structure $\mathcal{A}$, φ_{pre} will hold just if $P_1^{\mathcal{A}}$ is a reflexive, transitive relation.

So we want a formula ψ expressing some property of $\mathbb{N}$ and $\mathbb{R}$ that is not shared by all pre-orders, e.g. the fact that they are total orders. So take ψ to express the totality condition,

$\psi :=$ "$\forall x_1, x_2 \; P_1(x_1, x_2) \vee P_1(x_2, x_1)$".

This holds in $\mathbb{N}$ and $\mathbb{R}$ since, as noted, they're total. But any non-total partial order — e.g. $(\mathcal{P}(\mathbb{N}), \subseteq)$ — viewed as a structure,

$\langle \mathcal{P}(\mathbb{N}); \subseteq; \rangle$, gives an interpretation in which φ_{pre} holds and ψ fails.

(Failure of totality in $\mathcal{P}(\mathbb{N})$ is witnessed by e.g. $\{1\}, \{2\}$, since $\{1\} \nsubseteq \{2\}$, $\{2\} \nsubseteq \{1\}$.)

By soundness, this countermodel shows $\varphi_{pre} \nvdash \psi$.

11. (a) Suppose Γ, Δ are ded. closed; we will show that $\Gamma \cap \Delta$ also is.

If $\Gamma \cap \Delta \vdash \varphi$,

then since $\Gamma \cap \Delta \subseteq \Gamma$, we have $\Gamma \vdash \varphi$,

so by deductive closure, $\varphi \in \Gamma$;

and similarly, $\varphi \in \Delta$.

So $\varphi \in \Gamma \cap \Delta$, as desired.

(b) Let V_1, V_2 be two different interpretations, say with $p_1^{V_1} = 1$, $p_1^{V_2} = 0$.

Then the theories $Th(V_i) = \{\varphi \mid V_i \models \varphi\}$ are complete (as we saw in class):

for any φ, if $[\![\varphi]\!]^{V_i} = 1$ then $\varphi \in Th(V_i)$

and so $Th(V_i) \vdash \varphi$,

otherwise if $[\![\varphi]\!]^{V_i} = 0$ then $[\![\neg\varphi]\!]^{V_i} = 1$

& so $Th(V_i) \vdash \neg\varphi$.

But $Th(V_1) \cap Th(V_2)$ is not complete:

since $Th(V_1) \cap Th(V_2) \subseteq Th(V_i)$,

we have $V_i \models Th(V_1) \cap Th(V_2)$, for $i = 1, 2$,

so by soundness, $Th(V_1) \cap Th(V_2) \nvdash P_1$

(as $V_2 \nvDash P_1$)

& also $Th(V_1) \cap Th(V_2) \nvdash \neg P_1$

(as $V_1 \nvDash \neg P_1$).

(c) The maximal theory Form (all formulas)

is clearly complete (for any φ, $Form \vdash \varphi$

and $Form \vdash \neg\varphi$)

& deductively closed (for any φ, $\varphi \in Form$)

but is not consistent (since $\perp \in Form$)

so is not max. cons.