Algebraic Topology, VT22.
Homework Assignment 3.
Due Thursday February 10.

(1) (4 points) Let C_* and D_* be chain complexes. A map f of degree $n \in \mathbb{Z}$ from C_* to D_* is a sequence of homomorphisms $f_k \colon C_k \to D_{k+n}$ for $k \in \mathbb{Z}$. For a map f of degree n, define $\partial(f)$ to be the map of degree $n-1$ given by

$$\partial(f) = \partial^D \circ f - (-1)^n f \circ \partial^C,$$

where ∂^C and ∂^D are the boundary homomorphisms of C_* and D_*, respectively.

(a) Show that this makes

$$\ldots \to \mathrm{Hom}(C_*, D_*)_n \xrightarrow{\partial} \mathrm{Hom}(C_*, D_*)_{n-1} \to \ldots$$

into a chain complex, where $\mathrm{Hom}(C_*, D_*)_n$ is the set of all maps of degree n.

(b) Find an interpretation of $H_0\big(\mathrm{Hom}(C_*, D_*)\big)$.

(2) (3 points) Let C_* and D_* be chain complexes. Prove that the relation of being chain homotopic is an equivalence relation on the set of chain maps from C_* to D_*.

(3) (3 points) Using the increasing coordinates for the $(n+1)$-simplex,

$$\Delta^{n+1} = \big\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid 0 \le x_0 \le \ldots \le x_n \le 1\big\},$$

show that

$$\Delta^n \times I = \bigcup_{i=0}^{n} \Delta_i^{n+1},$$

where Δ_i^{n+1} is the image of the map $\eta_i \colon \Delta^{n+1} \to \Delta^n \times I$ defined by

$$\eta_i(x_0, \ldots, x_n) = \big((x_0, \ldots, \widehat{x_i}, \ldots, x_n), x_i\big).$$

Draw the picture for $n = 2$.