Chain complexes

A *chain complex* C_* is a sequence of homomorphisms of abelian groups

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

such $\partial^2 = 0$ i.e. $\partial_n \circ \partial_{n+1} = 0 \quad \forall n \in \mathbb{Z}$.

Ex: $C_*(X)$ singular chain complex of a space X.

A *chain map* $F: C_* \to D_*$ is a sequence of homomorphisms $F_n: C_n \to D_n$ such that

$$\begin{tikzcd} C_n \arrow[r, "\partial_n^C"] \arrow[d, "F_n"'] & C_{n-1} \arrow[d, "F_{n-1}"] \\ D_n \arrow[r, "\partial_n^D"] & D_{n-1} \end{tikzcd}$$

commutes for all n

$F_{n-1} \circ \partial_n^C = \partial_n^D \circ F_n$ or

$F\partial = \partial F$

Ex: $f: X \to Y$ cont. map $\Rightarrow$ $f_\#: C_*(X) \to C_*(Y)$

chain map

Every chain map $F: C_* \to D_*$ induces a homomorphism in homology $F_*: H_n(C) \to H_n(D)$.

Recall: $H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

$x \in \ker \partial_n$; notation $[x] = x + \operatorname{im} \partial_{n+1} \in H_n(C)$.
n-cycle

$F_*[x] = [F(x)]$.

Chain homotopy

$F, G : C_* \to D_*$ chain maps.

Def. A chain homotopy from F to G is a sequence of homomorphisms $s_n : C_n \to D_{n+1}$ such that

$$F - G = \partial s + s \partial$$

(i.e. $F_n - G_n = \partial_{n+1} \circ s_n + s_{n-1} \circ \partial_n$)

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "s_n"'] \arrow[d, shift right, "F_n"'] \arrow[d, shift left, "G_n"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1}
\end{tikzcd}$$

In this case, write $F \simeq G$ if such an s exists.

Exercise: $\simeq$ is an equivalence relation.

Proposition: If $F \simeq G : C_* \to D_*$, then

$F_* = G_* : H_n(C) \longrightarrow H_n(D)$ for all n.

Proof: Let $x \in \ker \partial_n$ be an n-cycle in C_n.

Then $F(x) - G(x) = \partial(s(x)) + s(\underbrace{\partial(x)}_{=0})$

$\Downarrow$

so

$F_*[x] = [F(x)] = [G(x)] = G_*[x]$. $\square$

Theorem $f, g : X \to Y$ cont. maps.

If $f \simeq g$, then $f_\# \simeq g_\# : C_*(X) \to C_*(Y)$.

In particular, $f_* = g_* : H_n(X) \to H_n(Y)$ for all n.

Corollary: If $X \simeq Y$ (homotopy ~~equivalence~~)

then $H_n(X) \cong H_n(Y)$ for all n.

Proof Cor: $X \simeq Y$ means that there are maps

$$X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y \quad \text{such that} \quad \beta \circ \alpha \simeq 1_X$$

$$\text{and} \quad \alpha \circ \beta \simeq 1_Y$$

By functoriality and the above theorem, this implies

$$\beta_* \circ \alpha_* = (\beta \circ \alpha)_* \underset{\text{by Thm}}{=} (1_X)_* = 1_{H_n(X)} : H_n(X) \to H_n(X)$$

and similarly $\alpha_* \circ \beta_* = 1_{H_n(Y)}$.

so α_* and β_* are isomorphisms and $\alpha_* = \beta_*^{-1}$.

$\square$

<u>Proof of Theorem</u> Let $H : X \times I \to Y$ be a homotopy from f to g, i.e., $H(x,0) = f(x)$

$(I = [0,1])$ $H(x,1) = g(x)$

$\forall x \in X$.

We want to construct $s_n : C_n(X) \to C_{n+1}(Y)$ such that $f_\# - g_\# = \partial s + s\partial$.

$\Big(S_n(X) =$ set of singular n-simplices, i.e., cont. maps $\Delta^n \longrightarrow X \Big)$

We will construct functions

$$h_i : S_n(X) \longrightarrow S_{n+1}(Y) \qquad 0 \le i \le n$$

that satisfy

$$\left.\begin{aligned}
d_0 h_0 &= f_\# \\
d_i h_j &= h_{j-1} d_i && i<j \\
d_j h_j &= d_j h_{j-1} && (i=j) \\
d_i h_j &= h_j d_{i-1} && i>j+1 \\
d_{n+1} h_n &= g_\#
\end{aligned}\right\} \quad (*)$$

as functions $S_n(X) \longrightarrow S_n(Y)$.

(Recall: $C_n(X) = \mathbb{Z} S_n(X)$ free abelian group)

Assuming we have this, define

$$s_n = \sum_{i=0}^{n} (-1)^i h_i : C_n(X) \to C_{n+1}(Y)$$

<u>Claim</u>: $f_\# - g_\# = \partial s + s\partial$ holds.

<u>Proof</u>: Expand the right hand side:

$$\sum_{i=0}^{n+1} \sum_{j=0}^{n} (-1)^{i+j} d_i h_j + \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} h_i d_j$$

Use the above equations and get lots of cancellations; what remains will be $f_\# - g_\#$.

$\square$

Intermission: increasing coordinates for $\underline{\Delta}^n$

we defined

$$\underline{\Delta}^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} 0 \le t_i \le 1 \\ \sum_{i=0}^{n} t_i = 1 \end{array} \right\}$$

barycentric coordinates

It will be useful to have another coordinate system; the "increasing coordinates". Write

$$x_0 = 0$$
$$x_1 = t_0$$
$$x_2 = t_0 + t_1$$
$$\vdots$$
$$x_n = t_0 + \dots + t_{n-1}$$
$$x_{n+1} = t_0 + \dots + t_n = 1$$

then

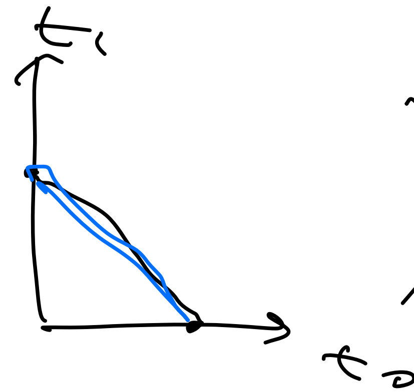
$$0 = x_0 \le x_1 \le x_2 \le \dots \le x_n \le x_{n+1} = 1$$

You can check that this defines a homeomorphism between Δ^n and

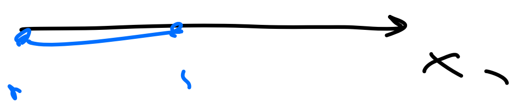
$$\Delta^n_x = \{(x_0, x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+2} \mid 0 = x_0 \le x_1 \le \dots \le x_n \le x_{n+1} = 1\}$$

$$\cong \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \le x_1 \le \dots \le x_n \le 1\}$$

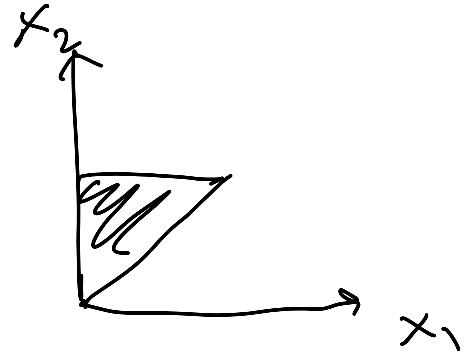
<u>Ex:</u> $\Delta^1_x = [0,1]$ $\quad \Big(\Delta^1$



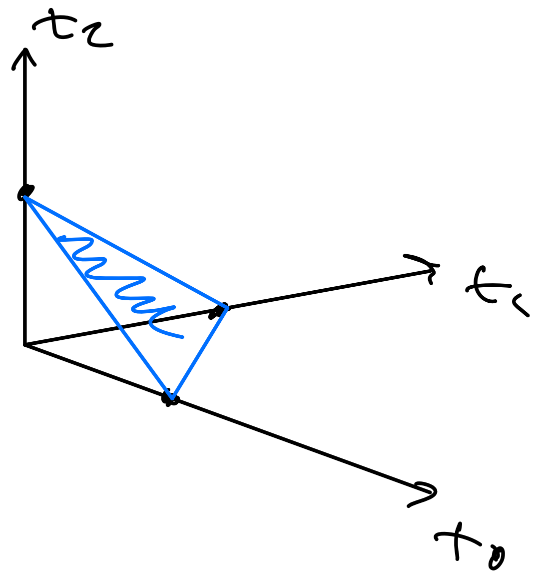
$\Big)$



Δ^2_x



Δ^2



Let's agree to use the increasing coordinates from now until the end of the lecture

$$\Delta^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \le x_1 \le \dots \le x_n \le 1\}$$

$$\begin{array}{c} S_n(X) = \{\text{cont. maps } \Delta^n \xrightarrow{\sigma} X\} \\ {\scriptstyle d_i}\big\downarrow \\ S_{n-1}(X) \end{array}$$

$x_0 = 0 \qquad x_n = 1$

$$d_i(\sigma)(x_0, \dots, x_n) = \sigma(x_0, \dots, x_i, x_i, \dots, x_n)$$

$$0 \le i \le n$$

<u>Construction of h_i</u>

$$\Delta^{n+1} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \overset{x_{-1}}{\overset{\shortparallel}{0}} \le x_0 \le \dots \le x_n \le \overset{x_{n+1}}{\overset{\shortparallel}{1}}\}$$

Define $h_i : S_n(X) \to S_{n+1}(Y)$ by:

for $\sigma : \Delta^n \to X$ $\quad h_i(\sigma) : \Delta^{n+1} \to Y$ is given by:

$$h_i(\sigma)(x_0, \dots, x_n) = H(\sigma(x_0, \dots, \hat{x}_i, \dots, x_n), x_i) \longleftarrow$$

(Recall: $H : X \times I \to Y$ homotopy from f to g)

Exercise: verify the identities (*)

Remark: The meaning behind this formula:

We can use the increasing coordinates to decompose $\Delta^n \times I$ as a union of $(n+1)$-dimensional simplices:

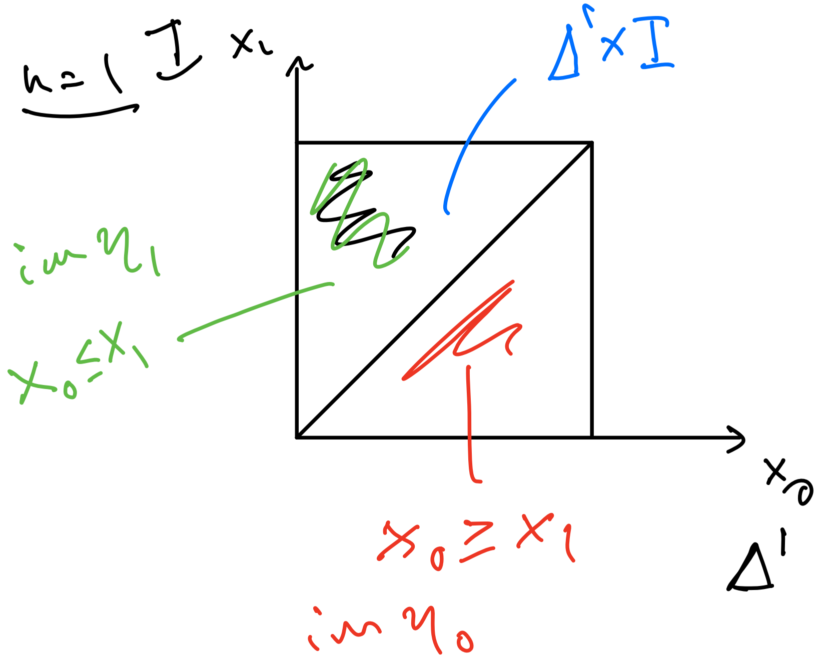
write $\eta_i : \Delta^{n+1} \longrightarrow \Delta^n \times I$

$$\eta_i(x_0, \dots, x_n) = \left((x_0, \dots, \hat{x}_i, \dots, x_n), x_i \right)$$

$$\Delta^n \times I = \bigcup_{i=0}^{n} \operatorname{im}(\eta_i)$$

$\Delta^2 = \{ (x_0, x_1) \mid 0 \le x_0 \le x_1 \le 1$

$\underline{n=1}$



$$\eta_0 : \Delta^2 \longrightarrow \Delta^1 \times I$$

$$\eta_0(x_0, x_1) = (x_1, x_0)$$

$$\eta_1(x_0, x_1) = (x_0, x_1)$$

$$\Delta^{n+1} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid 0 \le x_0 \le \dots \le x_n \le 1\}$$

$$\begin{array}{l} S_{n+1}(X) = \{\text{maps } \Delta^{n+1} \xrightarrow{\sigma} X\} \\ \quad \downarrow d_i \\ S_n(X) = \{\text{maps } \Delta^n \to X\} \end{array}$$

$$d_i(\sigma)(x_0, \dots, x_{n-1}) = \begin{cases} \sigma(0, x_0, \dots, x_{n-1}), & i = 0 \\ \sigma(x_0, \dots, x_{i-1}, x_{i-1}, \dots, x_{n-1}) & 0 < i < n+1 \\ \sigma(x_0, \dots, x_{n-1}, 1) & i = n+1 \end{cases}$$

if we write $x_{-1} = 0$ and $x_n = 1$ then this can be written

$$d_i(\sigma)(x_{-1}, \dots, x_n) = \sigma(x_{-1}, \dots, x_{i-1}, x_{i-1}, \dots, x_n)$$

Applications of homotopy invariance

We will show soon that $H_k(S^n) \cong \begin{cases} \mathbb{Z}, & k = 0, n \\ 0, & k \neq 0, n \end{cases}$

Assuming this we get

Corollary: $S^n \not\simeq S^m$ if $n \neq m$.

Corollary: $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ if $n \neq m$.

Proof: (We can't use homology directly, because

$\mathbb{R}^n \simeq * \simeq \mathbb{R}^m$ contractible,

so $H_k(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & k = 0 \\ 0, & k \neq 0 \end{cases}$)

Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a homeomorphism.
We may assume $f(0) = 0$.

then f restricts to a homeomorphism

$$f\big|_{\mathbb{R}^n\setminus\{0\}} : \mathbb{R}^n \setminus \{0\} \xrightarrow{\cong} \mathbb{R}^m \setminus \{0\}$$

Then use: $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\} \underset{f|}{\cong} \mathbb{R}^m \setminus \{0\} \simeq S^{m-1}$

so this would imply

$$S^{n-1} \simeq S^{m-1}$$

but this is impossible if $n \neq m$. $\square$

Remark: We have $X \simeq Y \Rightarrow H_n(X) \cong H_n(Y) \ \forall n$

What about the converse?

Answer: not true in general.

There are spaces X, Y such that $H_n(X) \cong H_n(Y) \ \forall n$ but $X \not\simeq Y$.

Ex: $X = \mathbb{C}P^2$, $Y = S^2 \vee S^4$

Exact sequences

A sequence of abelian groups

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow$$

is called exact if $\ker(\alpha_n) = \operatorname{im}(\alpha_{n+1})$ $\forall n \in \mathbb{Z}$

Remark: An exact sequence is the same thing as a chain complex A_* such that $H_n(A) \cong 0$ $\forall n \in \mathbb{Z}$:

$\operatorname{im}(\alpha_{n+1}) \subseteq \ker(\alpha_n) \iff \alpha_n \circ \alpha_{n+1} = 0$ chain complex

$\operatorname{im}(\alpha_{n+1}) = \ker(\alpha_n)$ exact sequence

holds if $H_n(A) = \ker(\alpha_n)/\operatorname{im}(\alpha_{n+1}) \cong 0$

Remark: If C_* is a chain complex, then $H_n(C)$ could be thought of as measuring the "failure of exactness".

Ex: $0 \to A \xrightarrow{\alpha} B$ exact $\Leftrightarrow$ α injective

$A \xrightarrow{\alpha} B \to 0$ exact $\Leftrightarrow$ α surjective

$0 \to A \xrightarrow{\alpha} B \to 0$ exact $\Leftrightarrow$ α isomorphism.

Post-lecture discussion starts here

$$s_n : C_n(X) \to C_{n+1}(Y)$$

$X \times I \xrightarrow{H} Y$ lets us produce

maps $\Delta^n \times I \to Y$ from singular simplices

$\Delta^n \xrightarrow{\sigma} X$. Indeed, just take

the composite

$$\Delta^n \times I \xrightarrow{\sigma \times 1} X \times I \xrightarrow{H} Y.$$

<u>but</u> we need some way to produce

$$\Delta^{n+1} \longrightarrow Y$$

from $\sigma : \Delta^n \to X$ and $H : X \times I \to Y$.

$$\Delta^{n+1} \xrightarrow{\eta_i} \Delta^n \times I \xrightarrow{\sigma \times 1} X \times I \xrightarrow{H} Y$$

Exercise in Hatcher

$F, G : C_* \to D_*$ are chain homotopic iff

$$\exists\, H : C_* \otimes I \longrightarrow D_* \qquad \begin{aligned} H(x \otimes (0)) &= F(x) \\ H(x \otimes (1)) &= G(x) \end{aligned}$$

I chain complex

$$\begin{tikzcd} 1 & & (01) \arrow[dl, "-1"'] \arrow[dr, "+1"] & \\ 0 & (0) & & (1) \end{tikzcd} \qquad \begin{aligned} \partial(01) &= (1) - (0) \\ \partial(0) = \partial(1) &= 0 \end{aligned}$$

$$\left(= \Delta_*(K) \qquad \begin{aligned} K_0 &= \{(0), (1)\} \\ K_1 &= \{(01)\} \end{aligned} \qquad \begin{aligned} d_0(01) &= (1) \\ d_1(01) &= (0) \end{aligned} \right)$$

$$\Delta_*(K) = I$$

$|K| \cong [0,1]$

$$(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q .$$

$$(C_* \otimes I)_n = C_n \otimes \mathbb{Z}\{(0),(1)\} \oplus C_{n-1} \otimes \mathbb{Z}\{(01)\}$$

$$\partial(x \otimes y) = \partial(x) \otimes y + (-1)^p x \otimes \partial(y).$$

$x \in C_p$