MT7041 – Markov chains and mixing times – exam

Date Tuesday February 8, 2022
Examiner Daniel Ahlberg
Tools None allowed.
Grading criteria The exam is divided into two parts, consisting of 20 and 40 points respectively. To pass the exam, a score of 14 or higher is required on Part I. If this is attained, then also Part II is corrected, and its score determines the grade. The grade corresponding to the scores on the different parts is determined by the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Each problem of Part I may give up to 5 points and every problem of Part II may give up to 10 points. Complete and well motivated solutions are required for full credit. Partial solutions may be awarded partial credit.

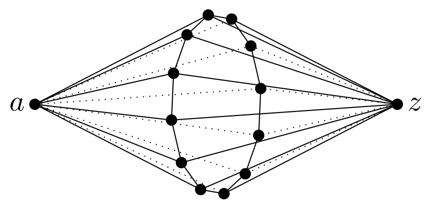
Part I

Problem 1. Let P be the transition matrix of a Markov chain on S. Show that P is irreducible if and only if for every $x, y \in S$ there exists $t \geq 1$ and a sequence $x_0, x_1, \dots, x_t$ such that $x_0 = x$, $x_t = y$ and $P(x_{i-1}, x_i) > 0$ for all $i = 1, 2, \dots, t$.

Problem 2. Let P be the transition matrix of an irreducible and aperiodic Markov chain on a finite state space S. Let π be the stationary distribution of P. Let μ be an arbitrary initial distribution and set $\nu = \frac{1}{2}(\mu + \pi)$. Show that for all $t \geq 1$

$$\|\nu P^t - \pi\|_{TV} = \frac{1}{2}\|\mu P^t - \pi\|_{TV}.$$

Problem 3. Consider the graph G obtained from a cycle of length n by adding two vertices a and z connected to all vertices of the cycle, but not between themselves. Compute the effective resistance between a and z of the network obtained by assigning unit conductances to all edges of the graph.



Problem 4. Let $(X_t)_{t\geq 0}$ be a Markov chain with transition matrix given by

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

Show that the mixing time t_{mix} of the chain equals 2.

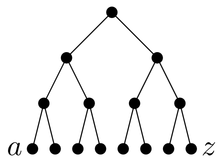
Part II

Problem 5. Recall that a *shuffle* is any Markov chain on the symmetric group $(S_n, \circ)$ with transition matrix of the form $P(\sigma, \rho\sigma) = \mu(\rho)$ for $\rho \in S_n$, where μ is some probability measure on S_n. Prove that

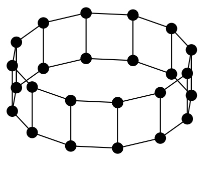
(a) if P is doubly stochastic, then the uniform distribution is stationary;

(b) P is doubly stochastic for any shuffle.

Problem 6. Consider a simple random walk on the *binary tree* of depth three (depicted below) started in the vertex labeled a. Determine the probability that the walker reaches the vertex labeled z before it returns to a.



Problem 7. Consider the graph (depicted below) obtained by aligning two cycles of length n and connecting vertices vertically. Construct a coalescing coupling of two simple random walks on the graph such that the two walkers are expected to meet in no more than Cn^2 steps, for some $C < \infty$. Motivate your answer (but you do not have to compute the expected coalescence time).



Problem 8. Let $Z_1, Z_2, \ldots$ be independent random variables that take on the values 0 and 1 with equal probability. Let $X_0 = x$, for some integer $x \geq 0$, and define recursively $X_{t+1} = 2X_t + Z_t$.

a) Show, for every $t \geq 1$, that X_t is uniformly distributed on the set
$$\{2^t x, 2^t x + 1, \ldots, 2^t x + 2^t - 1\}.$$

b) Let $Y_t := X_t \pmod{2^n}$. Determine the stationary distribution of $(Y_t)_{t\geq 0}$ and compute its mixing time t_{mix}.