MT7041 : Exam 2022-02-08 solutions

$\boxed{\text{PART I}}$

$\boxed{1}$ A transition matrix P is irreducible if $\forall x,y \in S$ $\exists t \geq 1$ such that

$$P^t(x,y) > 0.$$ 1p

Fix $x,y \in S$ and $t \geq 1$. The law of total probability gives that

$$\begin{aligned}
P^t(x,y) &= \mathbb{P}(X_t = y \mid X_0 = x) \\
&= \sum_{x_1,\dots,x_{t-1} \in S} \mathbb{P}(X_t = y \mid X_0 = x, X_1 = x_1, \dots, X_{t-1} = x_{t-1}) \cdot \mathbb{P}(X_1 = x_1, \dots, X_{t-1} = x_{t-1} \mid X_0 = x) \\
&= \sum_{x_1,\dots,x_{t-1} \in S} \mathbb{P}(X_t = y \mid X_0 = x, X_1 = x_1, \dots, X_{t-1} = x_{t-1}) \cdot \mathbb{P}(X_{t-1} = x_{t-1} \mid X_0 = x, \dots, X_{t-2} = x_{t-2}) \cdot \ldots \cdot \mathbb{P}(X_1 = x_1 \mid X_0 = x) \\
&\overset{\text{Markov prop}}{=} \sum_{x_1,\dots,x_{t-1} \in S} P(x,x_1) P(x_1,x_2) \cdots P(x_{t-1},y).
\end{aligned}$$ 2p

Since all terms in the sum are finite we conclude that

$$P^t(x,y) > 0 \iff \exists\, x_1, x_2, \dots, x_{t-1} \in S : P(x_{i-1}, x_i) > 0 \text{ for } i = 1,2,\dots,t \text{ and } x_0 = x,\ x_t = y.$$ 2p

This proves the claimed equivalence.

$\boxed{2}$ By proposition, the TV-distance can be written

$$\begin{aligned}
\| \nu P^t - \pi \|_{TV} &= \frac{1}{2} \sum_{y \in S} | \nu P^t(y) - \pi(y) | \quad \text{2p} \\
&= \frac{1}{2} \sum_{y \in S} \Big| \sum_{x \in S} \nu(x) P^t(x,y) - \pi(y) \Big| \\
&= \frac{1}{2} \sum_{y \in S} \Big| \frac{1}{2} \sum_{x \in S} \left[\mu(x) P^t(x,y) + \pi(x) P^t(x,y) \right] - \pi(y) \Big|.
\end{aligned}$$

Since π is stationary for P we have $\pi(y) = \sum_{x \in S} \pi(x) P^t(x,y)$ for $t \geq 1$. 2p
so the above reduces to

$$\frac{1}{2} \sum_{y \in S} \Big| \frac{1}{2} \mu P^t(y) - \frac{1}{2} \pi(y) \Big| = \frac{1}{2} \| \mu P^t - \pi \|_{TV}.$$ 1p

$\boxed{3}$ By definition the effective resistance is defined as

$$R(a \leftrightarrow z) := \frac{W(a) - W(z)}{\|I\|}.$$ 1p

where W is the (unit) voltage of the network and $\|I\| := \sum_{x \sim a} I(a,x)$ the strength of the current.

The unit voltage is the unique function that satisfies $W(a) = 1$, $W(z) = 0$ and is harmonic on $V \setminus \{a,z\}$. This implies that

$$W(x) = \tfrac{1}{2} \quad \forall x \in V \setminus \{a,z\}.$$ 2p

Now, either merge all nodes with equal voltage and use the network reduction laws. Or, compute the current directly (via Ohm's law)

$$I(a,x) := c(a,x) \cdot [W(a) - W(x)] = 1/2 \qquad \forall x \neq a,z.$$

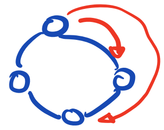
Hence $$R(a \leftrightarrow z) = \frac{1}{\|I\|} = \frac{1}{n \cdot 1/2} = \frac{2}{n}.$$ 2p

$\boxed{4}$ The mixing time is defined as the least $t \geq 1$ such that $d(t) \leq 1/4$, where

$$d(t) := \max_{x \in S} \| P^t(x, \cdot) - \pi \|_{TV}$$

and π is the stationary distribution of the chain. We thus need to verify that $d(1) > 1/4$ and $d(2) \leq 1/4$. 2p

First, we note that P is doubly stochastic (both columns and rows sum up to 1), so the uniform distribution is stationary for P. We also note that P is irreducible, so there are no other stationary distr. 1p



The chain corresponds to a RW on a cycle of length 4 where in each step the walker takes one or two steps clockwise. Hence, all states are equal and

$$d(1) = \|P(1,\cdot) - \pi\|_{TV} = \tfrac{1}{2} \sum_j |P(1,j) - \pi(j)| = 1/2$$

Moreover,

$$P^2(1,3) = P(1,2) \cdot P(2,3) = \tfrac{1}{2} \cdot \tfrac{1}{2} = 1/4 \qquad P^2(1,1) = 1/4$$

$$P^2(1,4) = P(1,2)P(2,4) + P(1,3)P(3,4) = 1/2 \qquad P^2(1,2) = 0$$ 2p

It follows that $d(2) = 1/4$, and $t_{mix} = 2$.

5 (a) Suppose that P is a doubly stochastic transition matrix on S, and let π denote the uniform distribution on S. Then,

$$\sum_{x\in S} \pi(x)P(x,y) = \frac{1}{|S|}\sum_{x\in S} P(x,y) = \frac{1}{|S|} = \pi(y).$$

So π is stationary for P. 3p

(b) Let P be the transition matrix of some shuffle. Rows sum to 1 by definition. We consider columns. For any pair $\sigma,\sigma' \in S_n$ there is a unique permutation $\rho = \sigma' \circ \sigma^{-1}$ that takes maps σ to σ'. This leads to

$$\sum_{\sigma\in S_n} P(\sigma,\sigma') = \sum_{\sigma\in S_n} P(\sigma,(\sigma'\circ\sigma^{-1})\circ\sigma) = \sum_{\sigma\in S_n} \mu(\sigma'\circ\sigma^{-1}).$$ 4p

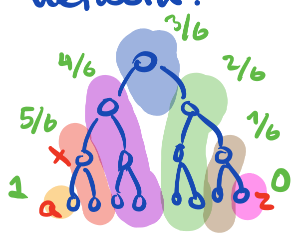
Since different σ in S_n have different inverse elements, as σ ranges over S_n so does σ^{-1} and thus $\sigma'\circ\sigma^{-1}$. It follows that

$$\sum_{\sigma\in S_n} P(\sigma,\sigma') = \sum_{\rho\in S_n} \mu(\rho) = 1,$$

since μ is a probability measure. Hence P is doubly stochastic. 3p

6 We may transform the graph into a network by assigning unit conductances to the edges of the graph. The network RW then corresponds to the SRW on the graph. 2p

By results from the network theory we have the following connection between the RW and the effective resistance of the network:



$$\mathbb{P}_a(\tau_z < \tau_a^+) = \frac{1}{c(a)\cdot R(a\leftrightarrow z)}.$$

Alternatively, since the walker from a necessarily jumps to x, we have

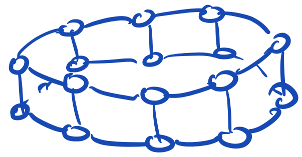
$$\mathbb{P}_a(\tau_z < \tau_a^+) = \mathbb{P}_x(\tau_z < \tau_a) = 1 - W(x),$$ 2p

where W is the (unit) voltage of the network. The voltage is the unique function which is $W(z)=0$, $W(a)=1$ and is harmonic on $V\setminus\{a,z\}$. The voltage is therefore constant on the colored regions, and with values given. The answer thus is $1/6$. 4p

2p

$\boxed{7}$

We construct a coupling of two lazy SRWs on the graph.



Set $X_0 = (x_1, y_1)$ and $Y_0 = (x_2, y_2)$. We construct the coupling so that the x-coordinates of the walkers align first, and then align the y-coordinates. 2p

Let $Z_1, Z_2, \ldots$ be independent Bernoulli (1/2). If the x-coordinates of X_{t-1} and Y_{t-1} differ, then

- if $Z_t = 1$ let $Y_t = Y_{t-1}$ and pick a neighbour v of X_{t-1} uniformly at random and set $X_t = v$;
- if $Z_t = 0$ let $X_t = X_{t-1}$ and pick a neighbour v of Y_{t-1} uniformly at random and set $Y_t = v$. 2p

If the x-coordinates of the two walkers coincide, but the y-coordinates differ, then we couple the walkers so that they jump horizontally together and vertically while the other rests. More precisely,

- with probab 1/6 move both chains left and with probab 1/6 move both chains right;
- with probab 1/3 let Z_t decide which chain jumps vertically and rest the other;
- with remaining probab 1/3 let both chains rest. 2p

Finally, if both chains coincide, then move them together.

In the first stage of the coupling the difference between the x-coordinates perform a lazy SRW on a single cycle. The expected time to hit zero is no more than $C \cdot n^2$ for some $C < \infty$. 2p

In the second stage of the coupling we wait for one of the chains to make a vertical jump. This time is geometrically distributed. This adds a constant to the expected coupling time. 2p

8 (a) For $t=1$ the statement says that X_1 takes the values $2x$ and $2x+1$ with equal probability. Since $X_1 = 2x + Z_1$ this is true by definition of Z_1.

We proceed by induction. So, suppose true for $t=m$. Then, for $k = 0, 1, \ldots, 2^{m+1}-1$ we have 2p

$$\mathbb{P}(X_{m+1} = 2^{m+1}x + k) = \sum_{\ell=0}^{2^m-1} \mathbb{P}(X_{m+1} = 2^{m+1}x + k \mid X_m = 2^m x + \ell) \cdot \mathbb{P}(X_m = 2^m x + \ell)$$

$$= \sum_{\ell=0}^{2^m-1} \mathbb{P}(Z_{m+1} = 2^{m+1}x + k - 2(2^m x + \ell)) \cdot \frac{1}{2^m}$$

$$= \sum_{\ell=0}^{2^m-1} \underbrace{\mathbb{P}(Z_{m+1} = k - 2\ell)}_{= \begin{cases} \frac{1}{2} & \text{if } \ell = \lfloor k/2 \rfloor \\ 0 & \text{otw} \end{cases}} \cdot \frac{1}{2^m} = \frac{1}{2^{m+1}}$$

2p

Hence, due to induction, the statement remains true for $t = m+1$ as required.

(b) The sequence $(Y_t)_{t\geq 0}$ takes values in $S = \mathbb{Z}_{2^n}$. Let

$$A_t = \{2^t x, 2^t x + 1, \ldots, 2^t x + 2^t - 1\}.$$

By (a) we conclude that Y_t is uniformly distributed on A_t (mod 2^n). More precisely, we have for $t < n$ that

$$P^t(x,y) = \begin{cases} \frac{1}{2^t} & y \in A_t \pmod{n} \\ 0 & \text{otw} \end{cases}$$

and for $t \geq n$

$$P^t(x,y) = \frac{1}{2^n} \qquad \text{for all } y \in \mathbb{Z}_{2^n}.$$ 2p

That is, $P^t(x,\cdot)$ coincides with the uniform distribution on $\mathbb{Z}_{2^n}$ for $t \geq n$. As a consequence, P is both irreducible and aperiodic. Since the chain converges to the uniform distribution, by the convergence theorem, that is its stationary distribution. 2p

We finally compute $d(t)$. From above we get

$$d(t) = \max_{x\in S} \| P^t(x,\cdot) - \pi \|_{TV}$$

$$= \max_{x \in S} \sum_{y \in A_t} \left| P^t(x,y) - \frac{1}{2^n} \right|$$

$$= 2^t \left(\frac{1}{2^t} - \frac{1}{2^n} \right) = 1 - 2^{-(n-t)} \qquad \text{for } t \leq n.$$

In particular, $d(n-1) = 1/2$ and $d(n) = 0$, so $t_{mix} = n$. 2p