Relative homology

X space. $A \subseteq X$ subspace.
Let $i : A \to X$ be the inclusion.

Then $i_\# : C_n(A) \to C_n(X)$ is injective $\forall n$.

Def. $C_n(X, A) = C_n(X)/C_n(A)$.

Since $i_\# : C_*(A) \to C_*(X)$ is a chain map,
we get an induced boundary homomorphism

$$\partial : C_n(X, A) \to C_{n-1}(X, A)$$

$$\partial(c + C_n(A)) = \partial(c) + C_{n-1}(A)$$

for $c \in C_n(X)$.

and chain complex $C_*(X, A)$.

Def. $H_n(X, A) = H_n(C_*(X, A))$.

$\underline{Q}$: What is the relation between $H_*(A)$, $H_*(X)$ and $H_*(X, A)$?

Naive guess: $H_n(X, A) \overset{?}{=} H_n(X)/H_n(A)$??

WRONG!

E.g. $A = S^1 \subset X = \mathbb{R}^2$

$H_1(X) = 0$ (X contractible)

$H_1(A) \cong \mathbb{Z}$

so $H_1(A) \to H_1(X)$ is not even injective...

Theorem There is a long exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{j_*} H_{n-1}(X,A) \to \cdots$$

$$\cdots \to H_0(A) \to H_0(X) \to H_0(X,A) \to 0$$

Remark: If we had $H_n(X,A) = H_n(X)/H_n(A)$, it would mean we had exact sequences

$$0 \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X)/H_n(A) \to 0$$

$$0 \to H_{n-1}(A) \to H_{n-1}(X) \to H_{n-1}(X)/H_{n-1}(A) \to 0$$

$$\vdots$$

Corollary: If $H_n(X,A) \cong 0$ for all n, then $H_n(A) \xrightarrow{i_*} H_n(X)$ is an isomorphism for all n.

Pf: From the long exact sequence, we get

$$\underbrace{H_{n+1}(X,A)}_{=0} \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} \underbrace{H_n(X,A)}_{=0}$$

$\Rightarrow$ i_* is omorphism. $\square$

In fact, the converse is true!
If i_* is an iso. for all n, then $H_n(X,A) = 0$ $\forall n$.

Reason: look at

$$H_n(A) \xrightarrow[\cong]{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow[\cong]{i_*} H_{n-1}(X)$$

Exactness means:

$\ker j_* = \operatorname{im} i_* = H_n(X) \Rightarrow j_* = 0$

$\uparrow$ i_* surjective

Exactness: $\ker \partial = \operatorname{im} j_* = 0 \Rightarrow \partial$ injective

Exactness: $\ker i_* = \operatorname{im} \partial \Rightarrow \partial = 0$
$\ker i_* = 0$ (i_* injective)

∂ injective and $\partial = 0 \Rightarrow H_n(X, A) = 0$.

Remark: Thus, $H_*(X, A)$ measures the difference between $H_*(X)$ and $H_*(A)$, in a sense.

Warning: Knowing $H_*(A)$ and $H_*(X, A)$ does not always allow you to deduce what $H_*(X)$ is.
(But in favorable situations one can do this.)

Topological interpretation

Theorem If $A \subseteq X$ is a closed subspace and if A is a deformation retract of some neighborhood $V \subseteq X$, then

$$H_n(X,A) \cong H_n(X/A, *) \left(\cong \tilde{H}_n(X/A)\right)$$

Reminder: That $A \subseteq V$ is a deformation retract means that there is a continuous family $f_t : V \to V$, $t \in [0,1]$, such that

$$f_0(x) = x \qquad \text{for all } x \in V$$

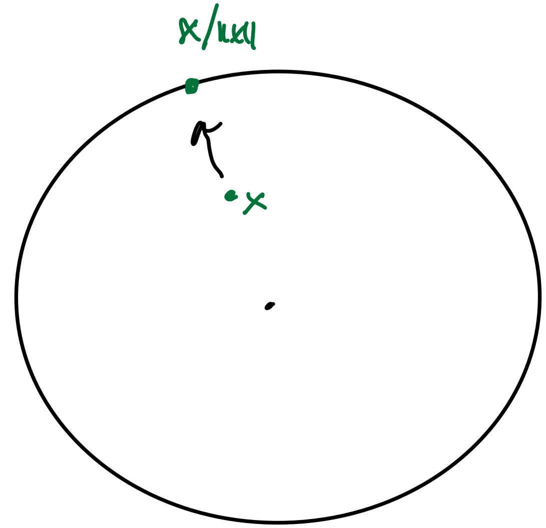
$$f_1(x) \in A \qquad \text{— " —}$$

$$f_t(a) = a \qquad \text{for all } a \in A \text{ and all } t \in [0,1].$$

Example: • $\underset{A}{S^{n-1}} \subseteq \underset{X}{D^n} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$

$V = D^n \setminus \{0\}$. Then $S^{n-1} \subseteq D^n \setminus \{0\}$ is a deformation retract: $f_t : D^n \setminus \{0\} \to D^n \setminus \{0\}$

$$f_t(x) = (1-t)\,x + t\,\frac{x}{\|x\|}$$



- More generally, if X is a Δ-complex and A is a subcomplex, then the hypothesis is fulfilled.

Reduced homology $\qquad \tilde H_n(X) := H_n(\tilde C_*(X))$.

$$C_*(X) \qquad \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow 0 \to \cdots$$

$$\tilde C_*(X) \qquad \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon = \partial_0} \mathbb{Z} \longrightarrow 0$$

$$n_1 x_1 + \dots + n_k x_k \overset{\varepsilon}{\longmapsto} n_1 + \dots + n_k$$

$$\begin{cases} \tilde H_n(X) = H_n(X) & n>0 \\ H_0(X) \cong \mathbb{Z} \oplus \tilde H_0(X) & \text{if } X \neq \emptyset \end{cases}$$

$$\boxed{\varepsilon \partial_1 = 0}$$

Computation of $H_*(S^n)$

Theorem: $\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, & k=n \\ 0, & k \neq n \end{cases} \quad \forall n \geq 0$

Proof: induction on n. $\quad S^0 = \{a\} \sqcup \{b\}$

$n=0$ $\quad \tilde{H}_k(S^0) = \begin{cases} \mathbb{Z}, & k=0 \\ 0, & k \neq 0 \end{cases}$

Assume by induction that $\tilde{H}_k(S^{n-1}) = \begin{cases} \mathbb{Z}, & k=n-1 \\ 0, & \text{else} \end{cases}$

Then use above theorem with $\underset{A}{S^{n-1}} \subseteq \underset{X}{D^n}$.

$D^n / S^{n-1} \cong S^n$

Get long exact sequence

$$\underset{=0}{\tilde{H}_k(D^n)} \xrightarrow{j_*} \tilde{H}_k(D^n, S^{n-1}) \xrightarrow{\partial} \tilde{H}_{k-1}(S^{n-1}) \xrightarrow{i_*} \underset{=0}{\tilde{H}_{k-1}(D^n)}$$

$\tilde{H}_k(D^n, S^{n-1}) \cong \tilde{H}_k(D^n/S^{n-1}) \cong \tilde{H}_k(S^n)$ by thm.

D^n contractible

$\Rightarrow \partial$ isomorphism

$$\Rightarrow \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1}) = \begin{cases} \mathbb{Z}, & k-1 = n-1 \Leftrightarrow k = n \\ 0, & \text{else} \end{cases}$$

$\square$

The long exact sequence

Suppose we have a short exact sequence of chain complexes

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \to 0,$$

This means we have a short exact sequence for every $n \in \mathbb{Z}$

$$\begin{array}{ccccccccc}
0 & \to & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{j_n} & C_n & \to & 0 \\
 & & \partial\downarrow & & \partial\downarrow & & \partial\downarrow & & \\
0 & \to & A_{n-1} & \xrightarrow{i_{n-1}} & B_{n-1} & \xrightarrow{j_{n-1}} & C_{n-1} & \to & 0
\end{array}$$

commutes: $\partial i = i \partial$

$\partial j = j \partial$

Theorem There is a "connecting homomorphism"

$$\partial : H_n(C) \to H_{n-1}(A)$$

such that the sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "j_*"] & H_n(C) \arrow[dll, "\partial"'] \\
& H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "j_*"] & H_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

is exact.

Remark: Applying this to the exact sequence

$$0 \to C_*(A) \to C_*(X) \to C_*(X,A) \to 0$$

yields the exact sequence in the theorem we stated earlier.

The connecting homomorphism $\partial : H_n(C) \to H_{n-1}(A)$ is defined as follows:

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "\partial"'] & B_n \arrow[r, "j_n"] \arrow[d, "\partial"'] & C_n \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] & B_{n-1} \arrow[r, "j_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & b \arrow[r, mapsto, "j"] \arrow[d, mapsto, "\partial"] & c \arrow[d, mapsto, "\partial"] \\
a \arrow[r, mapsto, "i"] \arrow[d, mapsto, "\partial"'] & \partial(b) \arrow[r, mapsto, "j"] \arrow[d, mapsto, "\partial"] & 0 \\
\partial(a) \arrow[r, mapsto, "i"] & 0 &
\end{tikzcd}$$

(1) Pick a cycle $c \in C_n$ $(\partial c = 0)$ representing a homology class $[c] \in H_n(C)$.

(2) Pick $b \in B_n$ with $j_n(b) = c$ (j surj. by exactness)

(3) $\ker j = \operatorname{im} i \Rightarrow$ can find $a \in A_{n-1}$ with $i(a) = \partial(b)$

(4) a is a cycle!

$$i(\partial(a)) = \partial(\partial(b)) = 0$$

$\Rightarrow \partial(a) = 0$.
i injective

Then define $\partial[c] = [a]$.

Here, a is any element of A_{n-1} such that $i(a) = \partial(b)$, where b is any element of B_n such that $j(b) = c$.

<u>Need to check</u>:
- Does not depend on representative c.
- Does not depend on the choice of b
- ∂ is a homomorphism.

Once these are checked, we have ∂.

To prove the theorem, we need to check exactness at all places:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \to$$

$\ker j_* = \operatorname{im} i_*$ $\qquad$ $\ker i_* = \operatorname{im} \partial$

$\ker \partial = \operatorname{im} j_*$

Let's do $\ker \partial = \operatorname{im} j_*$:

$\ker \partial \subseteq \operatorname{im} j_*$: Suppose $\partial[c] = 0$

$[j(b'')]$

Want to find cycle $b'' \in B_n$ such that $j_*[b''] = [c]$

$$\begin{tikzcd} & b' \arrow[r, mapsto, "j"] \arrow[dd, bend right, "\partial"'] & 0 \\ a' \arrow[ur, mapsto, "i"] \arrow[d, "\partial"'] & b \arrow[r, mapsto, "j"] \arrow[d, "\partial"] & c \\ a \arrow[r, mapsto, "i"] & \partial(b) & \end{tikzcd}$$

$j(b) = c$

By def. $\partial[c] = [a]$. $\partial[c] = 0$ then means $a = \partial(a')$ for some a'. By commutativity $\partial(b') = \partial(b)$.

$\Rightarrow \partial(b - b') = \partial(b) - \partial(b') = 0$.

$\Rightarrow b'' := b - b'$ is a cycle. Moreover,

$$j(b'') = j(b) - \underbrace{j(b')}_{= j(i(a')) = 0} = c. \quad \text{so} \quad j_*[b''] = [j(b'')] = [c]$$

$\square$

$\underline{\operatorname{im} j_* \subseteq \ker \partial}$: suppose $[c] = j_*[b''] = [j(b'')]$ for some cycle $b'' \in B_n$.

Then let's use that $\partial[c]$ does not depend on the representative c, only on the homology class $[c]$:

$$\partial[c] = \partial[j(b'')]$$

Here, we can use b'' as "our b" in the def. of ∂:

$$\begin{tikzcd} & b = b'' \arrow[r, mapsto, "j"] \arrow[d, mapsto, "\partial"'] & j(b'') = c \\ a = 0 \arrow[r, mapsto, "i"] & 0 & \end{tikzcd}$$

so $a = 0$ for these choices.

$\Rightarrow \partial[c] = \partial[j(b'')] = [a] = 0$.

Hence, $\partial j_*[b''] = 0$ i.e. $\operatorname{im} j_* \subseteq \ker \partial$.

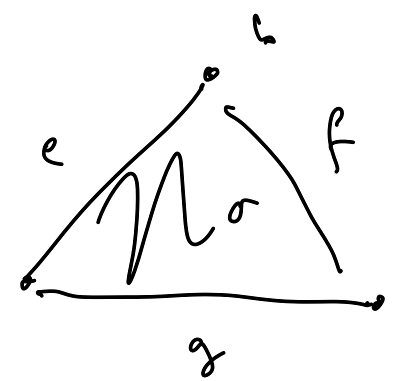
Let K be an abstract Δ-complex.

Def. A subcomplex $L \subseteq K$ is a collection of subsets

$$L_n \subseteq K_n \quad \text{such that} \quad d_i : K_n \longrightarrow K_{n-1}$$

satisfies $d_i(L_n) \subseteq L_{n-1}$. $\forall i, n$.

EX:



$K_0 = \{a, b, c\}$

$K_1 = \{e, f, g\}$

$K_2 = \{\sigma\}$

$|K| \cong D^2$

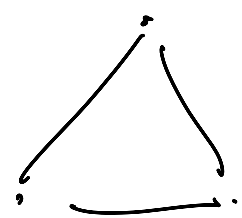
$\uparrow \qquad \cup|$

$|L| \cong S^1$

$L_0 = \{a, b, c\}$

$L_1 = \{e, f, g\}$

$L_2 = \emptyset$

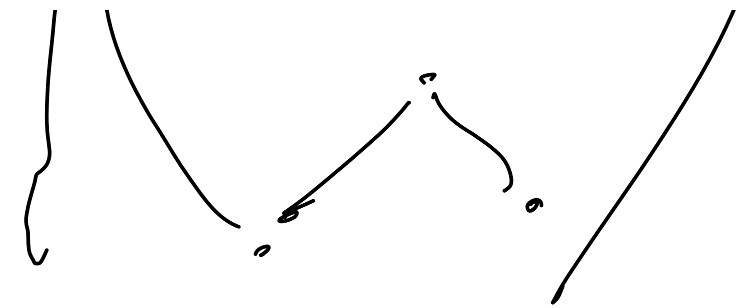


or

$L_0 = \{a, b, c\}$

$L_1 = \{e, f\}$

$L_2 = \emptyset$



Nonexample: $L_0 = \{c\}$

$L_1 = \{e\}$

$\leadsto \quad |L| \subseteq |K|$

$\quad\;\; \| \qquad\;\; \|$

$\quad\;\; A \subseteq X$

$D^n \cong \Delta^n$

$S^{n-1} = \partial D^n \cong \partial \Delta^n$