Algebraic Topology, VT22.
Homework Assignment 4.
Due Thursday February 17.

(1) (2 points) Find two non-isomorphic abelian groups G that fit in an exact sequence

$$0 \to \mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

(Bonus problem: show that every abelian group G as above is isomorphic to one of the two groups you found.)

(2) (4 points) Let X be a topological space and let $A \subseteq X$ be a simple closed curve, i.e., a subspace homeomorphic to the circle S^1. Suppose we know that

$$H_n(X, A) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

What can we deduce about the homology groups of X?

(3) (4 points) Let $f \colon A_* \to B_*$ be a chain map. Define the mapping cone $C(f)_*$ by

$$C(f)_n = A_{n-1} \oplus B_n,$$

and the boundary homomorphism $\partial \colon C(f)_n \to C(f)_{n-1}$ by

$$\partial(a, b) = (\partial(a), f(a) - \partial(b)).$$

(a) Verify that this makes $C(f)_*$ into a chain complex.
(b) Show that there is a short exact sequence of chain complexes

$$0 \to B_* \to C(f)_* \to A_{*-1} \to 0$$

and show that the connecting homomorphism may be identified with

$$\pm f_* \colon H_n(A) \to H_n(B).$$

(c) Use this to argue that f_* is an isomorphism if and only if all homology groups of $C(f)_*$ are trivial.