Algebraic Topology, VT22.
Homework Assignment 5.
Due Thursday February 24.

(1) (5 points) Let C_* be a chain complex and let $A_* \subseteq C_*$ be a subcomplex. Suppose that $S\colon C_* \to C_*$ is a chain map with the following properties:
  (a) For every $x \in C_*$, there is a non-negative integer m such that $S^m(x) \in A_*$.
  (b) There is a chain homotopy $T\colon C_* \to C_{*+1}$ from the identity to S,
$$1 - S = \partial T + T\partial.$$
  (c) Both S and T preserve A_*.

Prove that the homomorphism $H_*(A) \to H_*(C)$ induced by the inclusion is an isomorphism.

(2) (5 points) Recall the definition of the standard n-simplex:
$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \le t_i \le 1,\ t_0 + \ldots + t_n = 1\}.$$
For each permutation $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ of $0, 1, \ldots, n$, define
$$\Delta^n_\omega = \{(t_0, \ldots, t_n) \in \Delta^n \mid t_{\omega_0} \le \ldots \le t_{\omega_n}\}.$$
  (a) Show that Δ^n is the union of all subspaces of the form Δ^n_ω. Draw the picture for $n = 2$.
  (b) Find an explicit affine embedding
$$f_\omega\colon \Delta^n \to \Delta^n$$
  such that $\mathrm{im}(f_\omega) = \Delta^n_\omega$. Use this to argue that Δ^n_ω is homeomorphic to Δ^n.
  (c) (Bonus problem) Prove that for suitable choices of f_ω as above, the barycentric subdivision operator $S\colon C_*(X) \to C_*(X)$ is given by
$$S(\sigma) = \sum_\omega (-1)^\omega \sigma \circ f_\omega$$
  for singular simplices $\sigma\colon \Delta^n \to X$. The sum is over all permutations ω and $(-1)^\omega$ denotes the sign of the permutation.