THE COHOMOLOGY RING FOR REAL PROJECTIVE SPACE

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There are many ways to compute the cup product structure in the mod-2 cohomology of real projective spaces, $\mathbf{R}P^n$, but most of them are not directly accessible after you have learned the definition of the cup product. Hatcher uses the Künneth theorem and a bit of geometry. Other, more conceptual proofs use the Gysin sequence or Poincaré duality for manifolds. This note is to show how to compute the result using nothing but the combinatorial definition of the cup product on simplicial complexes, and a suitable simplicial complex structure on $\mathbf{R}P^n$. I assume as known that

$$H^j(\mathbf{R}P^n;\mathbf{Z}/2\mathbf{Z}) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z}; & 0 \leq j \leq n \\ 0; & \text{otherwise.} \end{cases}$$

Recall that $\mathbf{R}P^{n-1}$ can be thought of as the standard $(n-1)$-sphere with antipodal points identified. We give $\mathbf{R}P^{n-1}$ the *octahedral triangulation*:

Let e_i^+ be the ith standard basis vector of $\mathbf{R}^n$, and denote by e_i^- its negative, $-e_i^+$. (It will become clear in a moment why we want to avoid writing just e_i and $-e_i$.) The $(n-1)$-dimensional octahedron S_n is defined as the boundary of the complex hull of the $2n$ vectors $e_1^\pm, \dots, e_n^\pm$. This is an $(n-1)$-dimensional simplicial complex homeomorphic to the $(n-1)$-dimensional sphere, whose k-simplices are the complex hulls of sets of the form $\{e_{i_1}^{\epsilon_1}, e_{i_2}^{\epsilon_2}, \dots, e_{i_k}^{\epsilon_k}\}$, where $1 \leq i_1 < \cdots < i_k \leq n$ and $\epsilon_j \in \{+,-\}$. We will denote this simplex by $e_{i_1}^{\epsilon_1} \cdots e_{i_k}^{\epsilon_k}$. Thus there are total of $2^k\binom{n}{k}$ many k-simplices. The faces of a k-simplex are given by omitting one of the $e_{i_j}^{\epsilon_j}$-factors.

Note that S_n has an involution given by sending a point x to $-x$. In terms of simplices, it sends $e_{i_1}^{\epsilon_1} \cdots e_{i_k}^{\epsilon_k}$ to $e_{i_1}^{-\epsilon_1} \cdots e_{i_k}^{-\epsilon_k}$, so it respects the simplicial structure. Thus the quotient $P_n = S_n/\pm 1$ is a simplicial model for $\mathbf{R}P^{n-1} = \mathbf{S}^{n-1}/\pm 1$.

Consider the simplicial $(n-1)$-chain

$$[P_n] = \sum_{\epsilon \in \{\pm\}^{n-1}} e_1^{\epsilon_1} \cdots e_{n-1}^{\epsilon_{n-1}} e_n \in C_{n-1}(P_n).$$

This is a cycle modulo 2 because every $(n-1)$-simplex $e_1^\pm \cdots \widehat{e_k^\pm} \cdots e_{n-1}^\pm e_n$ occurs as a face of exactly two 2-simplices (those containing e_k^+ and those containing e_k^-). This is also true for the $(n-1)$-simplex $e_1^{\epsilon_1} \cdots e_{n-1}^{\epsilon_{n-1}}$, which is a face of $e_1^{\epsilon_1} \cdots e_{n-1}^{\epsilon_{n-1}} e_n$ and of $e_1^{-\epsilon_1} \cdots e_{n-1}^{-\epsilon_{n-1}} e_n$ by the identification of antipodal simplices. So $[P_n]$ is the unique nontrivial element in $H_{n-1}(P_n;\mathbf{Z}/2\mathbf{Z})$.

Now let us define a 1-cochain $\phi\colon C_1(P_n) \to \mathbf{Z}/2\mathbf{Z}$ as follows:

$$\phi(e_i^+ e_j^+) = \begin{cases} 1; & j-i \text{ odd} \\ 0; & j-i \text{ even;} \end{cases} \qquad \phi(e_i^- e_j^+) = \begin{cases} 0; & j-i \text{ odd} \\ 1; & j-i \text{ even.} \end{cases}$$

To see that this is a cocycle, we have to verify that it vanishes on the boundary of $e_i^\pm e_j^\pm e_k^\pm$, which is $e_i^\pm e_j^\pm - e_i^\pm e_k^\pm + e_j^\pm e_k^\pm$. (Signs do not matter since ϕ takes values in $\mathbf{Z}/2\mathbf{Z}$.) This is easy: if all the $\pm$ signs are equal then either $j-i$, $k-i$, and $k-j$ are all even ($\phi = 0$ on every summand), or exactly two of them are odd ($\phi = 1$ on exactly 2). Otherwise, two of the $\pm$ are the same and the third is different, let's say $e_i^+ e_j^+ e_k^-$. Then the same argument works for the numbers $\{i, j, k+1\}$.

Now, by the formula for the cup product, we have that

$$\phi^{n-1}([P_n]) = \sum_{\epsilon \in \{+,-\}^{n-1}} \phi(e_1^{\epsilon_1} e_2^{\epsilon_2}) \phi(e_2^{\epsilon_2} e_3^{\epsilon_3}) \cdots \phi_(e_{n-1}^{\epsilon_{n-1}} e_n).$$

By the definition of ϕ, it is nonzero on exactly one factor, namely when $\epsilon = (+, \dots, +)$. Thus

$$\phi^{n-1} \neq 0.$$

This shows immediately that $H^*(\mathbf{R}P^{n-1}; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}[\phi]/(\phi^n)$.