Algebraic Topology,
Homework Assignment 10,
Due Thursday April 7, 2022

(1) (5 points)
Let $i\colon A \hookrightarrow X$ be an inclusion of a subset, $i^*\colon H^*(X) \to H^*(A)$ the induced map in cohomology, and $d\colon H^*(A) \to H^{*+1}(X, A)$ the connecting homomorphism for the pair (X, A). Show that for $x \in H^*(X)$, $a \in H^*(A)$,

$$d(a \cup i^*(x)) = d(a) \cup x \in H^*(X, A).$$

(2) (5 points) For $n > 1$, Let $f\colon \mathbf{S}^{2n-1} \to \mathbf{S}^n$ be a map with mapping cone $C_f = \mathbf{S}^n \cup_f e^{2n}$, i.e. the two-cell complex where the $2n$-dimensional cell is attached to the n-dimensional cell by f. Then $H^0(C_f) \cong H^n(C_f) \cong H^{2n}(C_f) \cong \mathbf{Z}$ and all other cohomology groups are 0. Denote a generator of H^n by x and a generator of H^{2n} by y. Then $x \cup x = \alpha y$ for some $\alpha \in \mathbf{Z}$ which is well-defined up to sign. We call $|\alpha|$ the *Hopf invariant* of f and write $|\alpha| = h(f)$. Show: for every even n, there exists a map f with nonzero Hopf invariant. *Hint:* Consider a quotient space of $\mathbf{S}^n \times \mathbf{S}^n$.