MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Written examination in
Mathematics III – ODE
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1. **(4p)** Solve the initial value problem $\frac{\mathrm{d}u}{\mathrm{d}t} = e^u \sin t$, $u(0) = 0$.

   Solution: The DE has separate variables and the u-dependent factor never gets zero. Dividing by e^u and integrating yields

   $$\int e^{-u} \mathrm{d}u = \int \sin t \mathrm{d}t,$$

   equivalently $e^{-u} = \cos t + C$ with $C \in \mathbb{R}$. Thus $u = -\log(\cos t + C)$ is the general solution to the DE. To satisfy the initial condition we need

   $$0 = u(0) = -\log(1 + C),$$

   which has $C = 0$ as its only solution. Thus the unique solution to the BVP is

   $$u = -\log\left(\cos t\right).$$

   The maximal interval around the initial point $t_0 = 0$ where this solution is well-defined is $(-\pi/2, \pi/2)$.

2. **(6p)** Let $a \in \{1, \dots, 12\}$ be the number of your month of birth. (For instance, $a = 1$ if you are born in January, $a = 7$ if you are born in July, or $a = 10$ if you are born in October.) For your a, determine the general solution to the system

   $$\begin{cases} x' = -(a+2)x + (2a+2)y, \\ y' = -(a+1)x + (2a+1)y, \\ z' = (2a+2)x - (2a+2)y + az. \end{cases}$$

   Solution: We sketch a possible solution in dependence of the parameter a. The system is linear and homogeneous and may be written

   $$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} -(a+2) & 2a+2 & 0 \\ -(a+1) & 2a+1 & 0 \\ 2a+2 & -(2a+2) & a \end{pmatrix}.$$

   The characteristic polynomial of A is $p_A(\lambda) = (a-\lambda)(-a\lambda - a + \lambda^2 + \lambda)$, which leads to the eigenvalues a (with algebraic multiplicity 2) and -1 with multiplicity one. Computing the corresponding eigenvectors yields $(1,1,0)^\top$ and $(0,0,1)^\top$ for $\lambda = a$ and $(2,1,-2)^\top$ for $\lambda = -1$. In particular, A is diagonalizable. Therefore we may directly write down the general solution to the system. It is given by

   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix}(t) = c_1 e^{at} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{at} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

   with $c_1, c_2, c_3 \in \mathbb{R}$.

3. **(4p)** Determine all functions $y : \mathbb{R} \to \mathbb{R}$ such that y' and xy are linearly dependent.

Solution: y' and xy being linearly dependent means there exist a, b not both zero simultaneously such that $ay' + bxy = 0$ for all $x \in \mathbb{R}$. The case $a = 0, b \neq 0$ has only the trivial solution $y = 0$ identically. Therefore we may divide by a and get the DE

$$y' + cxy = 0.$$

This is a linear equation of first order and can be solved e.g. by multiplying the DE with the factor

$$e^{\int cx\mathrm{d}x} = e^{c\frac{x^2}{2}}.$$

This yields the DE

$$\left(e^{c\frac{x^2}{2}} y\right)' = 0,$$

i.e.

$$e^{c\frac{x^2}{2}} y = C, \quad C \in \mathbb{R}.$$

Hence the general solution is

$$y(x) = Ce^{-c\frac{x^2}{2}}, \quad c, C \in \mathbb{R}.$$

This is the whole class of functions such that y and xy are linearly dependent.

4. **(5p)** Use the **Laplace transform** to solve the initial value problem

$$u'' + 4u' + 3u = 12, \quad u(0) = 7, \quad u'(0) = 1.$$

Solution: Denoting by U the Laplace transform of u we get

$$s^2U(s) - su(0) - u'(0) + 4sU(s) - 4u(0) + 3U(s) = \frac{12}{s}.$$

After simplifying and plugging in the initial values we get the equation

$$s\big(s^2 + 4s + 3\big)U(s) - 7s(s + 4) - s = 12.$$

Its solution is

$$U(s) = \frac{7s^2 + 29s + 12}{s(s^2 + 4s + 3)} = \frac{a}{s(s + 3)(s + 1)}.$$

It remains to transform backwards. We use partial fractions to obtain

$$U(s) = \frac{5}{s + 1} - \frac{2}{s + 3} + \frac{4}{s}.$$

Hence

$$u(t) = 5e^{-t} - 2e^{-3t} + 4.$$

5. **(6p)** Let again a be the number of your month of birth. Rewrite the differential equation

$$x'' + ax' - x^2 + 1 = 0$$

as a first-order system and compute all its equilibrium points. Moreover, for each equilibrium point, investigate whether it is unstable/stable/asymptotically stable.

Solution: We again provide a sketch of a solution in dependence of the parameter a. To rewrite the equation as a system, let $y_1 = x$ and $y_2 = x'$. Then

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_1^2 - ay_2 - 1. \end{aligned}$$

Determining equilibrium points, the first equation gives $y_2 = 0$ and then the second one implies $y_1^2 = 1$. Hence the system has two equilibrium points, $(-1, 0)$ and $(1, 0)$. We study stability via linearization. Denoting the right-hand side of the system

$$F(y_1, y_2) = \begin{pmatrix} y_2 \\ y_1^2 - ay_2 - 1 \end{pmatrix}$$

we get the Jacobi matrix

$$F'(y_1, y_2) = \begin{pmatrix} 0 & 1 \\ 2y_1 & -a \end{pmatrix}.$$

The matrix

$$F'(-1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & -a \end{pmatrix}$$

has characteristic polynomial $\lambda^2 + a\lambda + 2$ and, thus eigenvalues

$$\lambda = -\frac{a}{2} \pm \sqrt{\frac{a^2 - 8}{4}} = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 8}\right).$$

For $a = 1, 2$ the expression under the square root is negative and therefore the real part of each eigenvalue is $-a/2 < 0$. For all other a both eigenvalues are real and negative, as $\sqrt{a^2 - 8} < a$. Therefore the critical point $(-1, 0)$ is asymptotically stable.

For the second equilibrium point we have

$$F'(1, 0) = \begin{pmatrix} 0 & 1 \\ 2 & -a \end{pmatrix},$$

and the characteristic polynomial is $\lambda(a + \lambda) - 2 = \lambda^2 + a\lambda - 2$. The eigenvalues are therefore

$$\lambda = \frac{1}{2}\left(-a \pm \sqrt{a^2 + 8}\right).$$

They are real and the larger one is always positive as $\sqrt{a^2 + 8} > a$. Hence the equilibrium point $(1, 0)$ is unstable.

6. **(5p)** Determine the general solution to the differential equation

$$y'' - y = xe^x.$$

Solution: The equation is linear and the corresponding homogeneous equation is $y'' - y = 0$ with characteristic polynomial $\lambda^2 - 1$. This leads to $\lambda = \pm 1$. Thus the general solution to the homogeneous equation is

$$y_{\text{hom}}(x) = C_1 e^x + C_2 e^{-x}, \quad C_1, C_2 \in \mathbb{R}.$$

To solve the inhomogeneous equation we use an ansatz $y_\mathrm{p}(x) = x(Ax+B)e^x$ – the factor x is due to resonance! Plugging this into the DE and collecting terms yields

$$(2B + 2A + 4Ax)e^x = xe^x.$$

Comparing coefficients leads to $2B + 2A = 0$ and $4A = 1$ and, thus, $A = 1/4, B = -1/4$. We obtain

$$y_\mathrm{p}(x) = x\left(-\frac{1}{4} + \frac{1}{4}x\right)e^x = \frac{1}{4}x(x-1)e^x.$$

Therefore the general solution to the inhomogeneous DE is

$$y(x) = C_1e^x + C_2e^{-x} + \frac{1}{4}x(x-1)e^x, \quad C_1, C_2 \in \mathbb{R}.$$