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Bonus test II
Mathematical Methods for Economists
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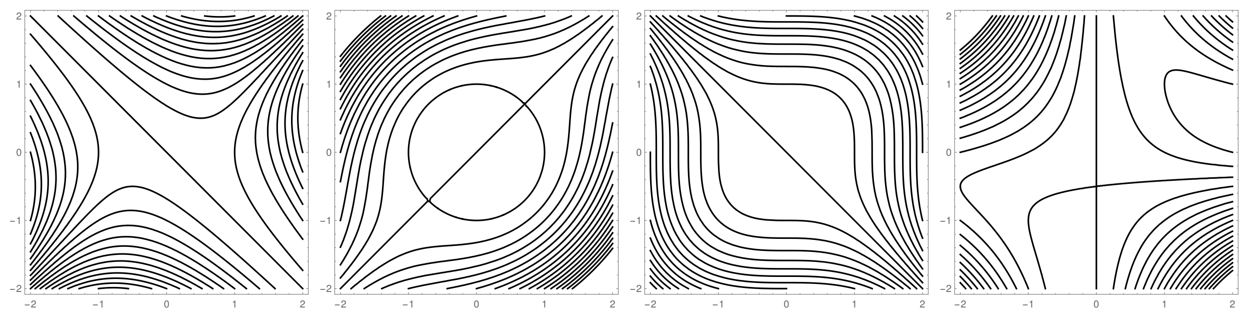
Problem 1. Let $f(x, y) = (x - y)(x^2 + y^2 - 1)$. Which of the following four figures shows the level curves of f?



Solution. There are many ways to solve this question. Since f is factored, let us examine the factors. The first factor, $(x - y)$ is 0 whenever $x = y$, so on the line $x = y$, the function has value 0. Hence, the line $y = x$ is a level curve for f. The only figure with $y = x$ as level curve is the second figure. To really verify the solution, we also note that the curve $x^2 + y^2 = 1$ is (with the same motivation) a level curve of f, so the unit circle is also a level curve of f. The second figure includes the unit circle as level curve.

Alternatively, we can compute the gradient of f at $(x, y) = (0, 0)$. With some work, one can compute that $(\nabla f)(0, 0) = (-1, 1)$, so the gradient vector is pointing in this direction. This is the same as $135°$. Now, the gradient is perpendicular to the level curve, so the level curve for f must have slope $45°$ in the origin. Only the second picture has this property.

Problem 2. Let $f(x, y) = x^2 - 2xy + 2y^3 - x - 3y$.

a) Compute the gradient of f in the point $(2, 3)$.

b) Compute the Hessian matrix for f.

c) Find the global minimum for f on the region $1 \leq x \leq 20$, $1 \leq y \leq 20$. Hint: prove that f is convex on this region[1].

Solution. a) We compute the partial derivatives, and get $f'_x = 2x - 2y - 1$ and $f'_y = -2x + 6y^2 - 3$. Thus, the gradient is $\nabla f = (2x - 2y - 1, -2x + 6y^2 - 3)$. In the point $(x, y) = (2, 3)$, this has value $(-3, 47)$.

b) Moving on to the second order derivatives, we get $f''_{xx} = 2$, $f''_{yy} = 12y$, $f''_{xy} = -2$. Thus, the Hessian matrix is $\begin{pmatrix} 2 & -2 \\ -2 & 12y \end{pmatrix}$.

[1]The original problem said $0 \leq x, y \leq 20$, and the function is not convex in this larger region. To find minimum for that problem, more work is needed.

c) We first look for all critical points in the region. A point (x, y) is critical if it makes the gradient $(0, 0)$, so we must solve

$$\begin{cases} 2x - 2y - 1 & = 0 \\ -2x + 6y^2 - 3 & = 0. \end{cases}$$

From the first equation, we see that $2x = 1 + 2y$. Substituting into the second gives $-(1 + 2y) + 6y^2 - 3 = 0$. This leads to $6y^2 - 2y - 4 = 0$ which has the solutions $y = 1$ and $y = -\frac{2}{3}$. The relation $2x = 1 + 2y$ allows us to compute the corresponding values for x. We get that $(\frac{3}{2}, 1)$ and $(-\frac{1}{6}, -\frac{2}{3})$ are critical points for f, but only the first point lies in the region. If we can prove that f is convex, we know that $(\frac{3}{2}, 1)$ is where the minimum is attained. In order for f to be convex, we need to verify the three conditions

$$f''_{xx} \geq 0, \qquad f''_{yy} \geq 0 \qquad \text{and} \qquad f''_{xx} f''_{yy} - (f''_{xy})^2 \geq 0.$$

In our situation, this leads to

$$2 \geq 0, \qquad 12y \geq 0 \qquad \text{and} \qquad 2 \cdot 12y - (-2)^2 \geq 0,$$

which is true in our region, since $y \geq 1$ there. Hence, the function is convex, so $f(\frac{3}{2}, 1) = -\frac{13}{4} = -3.25$ is the minimum.

Problem 3. An unknown function $f(x, y, z)$ has the partial derivatives

$$\frac{\partial f}{\partial x} = 2x(x+1)e^{2x+3y-z} \qquad \frac{\partial f}{\partial y} = 3x^2 e^{2x+3y-z} \qquad \frac{\partial f}{\partial z} = -x^2 e^{2x+3y-z}.$$

We let $g(s, t) = f\left(2s + 7t, 3s - 8t, 13s - 10t\right)$. Compute the partial derivative $\frac{\partial g}{\partial t}$.

Solution. The multivariate chain rule states that

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

We have $x(s, t) = 2s + 7t$, $y(s, t) = 3s - 8t$ and $z(s, t) = 13s - 10t$, so

$$\frac{\partial x}{\partial t} = 7, \qquad \frac{\partial y}{\partial t} = -8, \qquad \frac{\partial z}{\partial t} = -10.$$

Hence,

$$\begin{aligned} \frac{\partial g}{\partial t} &= 2x(x+1)e^{2x+3y-z} \cdot 7 \ + \ 3x^2 e^{2x+3y-z} \cdot (-8) \ + \ (-x^2)e^{2x+3y-z} \cdot (-10) \\ &= e^{2x+3y-z}\left(14x(x+1) - 24x^2 + 10x^2\right) \\ &= e^{2x+3y-z} \cdot 14x \end{aligned}$$

We now want to express the x, y, z in s, t. First, note that $e^{2x+3y-z}$ becomes

$$e^{2(2s+7t)+3(3s-8t)-(13s-10t)} = e^{4s+14t+9s-24t=13s-10t} = e^0 = 1,$$

so all in all $\frac{\partial g}{\partial t} = e^0 \cdot 14(2s + 7t) = 28s + 98t$.

Problem 4. Let $f(x,y) = 3x + 4y$. Find the maximum of f subject to the constraints $(x-1)^2 + y^2 \leq 25$.

Solution. The constraint $(x-1)^2 + y^2 \leq 25$ gives a region in the plane (it is a disk centered in $(1,0)$ with radius 5). The maximum is either attained inside the disk, or on the boundary. The partial derivatives are $f'_x = 3$, $f'_y = 4$, so f has no critical points. Hence, the maximum must be on the boundary on the region, which is given by the condition $(x-1)^2 + y^2 - 25 = 0$. We introduce the Lagrangian,

$$\mathcal{L}(x,y,\lambda) = 3x + 4y - \lambda((x-1)^2 + y^2 - 25).$$

We differentiate, and get

$$\mathcal{L}'_1(x,y) = 3 - 2\lambda(x-1) \qquad \mathcal{L}'_2(x,y) = 4 - 2\lambda y.$$

We must now solve the system

$$\begin{cases} 3 - 2\lambda(x-1) & = 0 \\ 4 - 2\lambda y & = 0 \\ (x-1)^2 + y^2 - 25 & = 0. \end{cases}$$

The first equation gives $x - 1 = \frac{3}{2\lambda}$, the second gives $y = 2/\lambda$. Substituting this into the third equation gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 25 = 0 \iff \frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 25 \iff \frac{9}{4} + \frac{16}{4} = 25\lambda^2 \iff \frac{25}{4} = 25\lambda^2.$$

This leads to $\lambda = \pm\frac{1}{2}$. We can then solve for x and y:

$$(\lambda, x, y) = (1/2, 4, 4) \qquad (\lambda, x, y) = (-1/2, -2, -4).$$

Thus, the maximum must be among the values $f(4,4) = 28$ and $f(-2,-4) = -22$, so the maximum of f on the disk is therefore 28.

Problem 5. Let A and B be the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 6 \\ 1 & 2 & -t \\ 1 & 1 & t \end{pmatrix}.$$

a) Compute $|A|$ and $|B|$.

b) For what values of t is the matrix B invertible?

c) Give an argument why there is no 2×2-matrix X (with real numbers), with the property that $X^2 = A$.

Solution. a) The formula for 2×2-determinats gives $|A| = 1 \cdot 3 - 5 \cdot 2 = -7$. Sarrus's rule for 3×3-determinants gives

$$|B| = 2t + 0 + 6 - (-t) - 0 - 12 = 3t - 6.$$

b) The matrix is invertible exactly when its determinant is non-zero, so we need $t \neq 2$.
c) Suppose $X^2 = A$. Then $|X|^2 = |A|$ by rules of determinants. But then, we must have that $|X|^2 = -7$. There is no real number whose square is -7, so there cannot be such a matrix X.

Problem 6. Solve the system of equations

$$\begin{cases} x \quad + \quad 6z &= 1 \\ x + 2y - tz &= 2 \\ x + y + tz &= 3, \end{cases}$$

under the assumption that $t \neq 2$.

Solution. We first write the system in matrix form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 6 & 1 \\ 1 & 2 & -t & 2 \\ 1 & 1 & t & 3 \end{array}\right).$$

We now perform Gaussian elimination, the first row is used to eliminate the other two 1s in the first column. We get (with more steps of Gaussian elimination),

$$\left(\begin{array}{ccc|c} 1 & 0 & 6 & 1 \\ 0 & 2 & -t-6 & 1 \\ 0 & 1 & t-6 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 6 & 1 \\ 0 & 1 & t-6 & 2 \\ 0 & 2 & -t-6 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 6 & 1 \\ 0 & 1 & t-6 & 2 \\ 0 & 0 & -3t+6 & -3 \end{array}\right) \tag{1}$$

From here, since $t \neq 2$, we can divide the last row by $-3t + 6$, and we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 6 & 1 \\ 0 & 1 & t-6 & 2 \\ 0 & 0 & 1 & \frac{1}{t-2} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 - \frac{6}{t-2} \\ 0 & 1 & 0 & 2 - \frac{t-6}{t-2} \\ 0 & 0 & 1 & \frac{1}{t-2} \end{array}\right).$$

Hence, under the condition $t \neq 2$, the solution is

$$x = 1 - \frac{6}{t-2} = \frac{t-8}{t-2}, \qquad y = 2 - \frac{t-6}{t-2} = \frac{t+2}{t-2}, \qquad z = \frac{1}{t-2}.$$

In the case $t = 2$, then the last line in the last matrix in (1) states $0 = -3$, which means that the system does not have any solutions in this situation.