Algebraic Topology,
Homework Assignment 13 (bonus)
Due Thursday May 5, 2022

(1) (5 points)
Let $\mathbf{S}^3 \subseteq \mathbf{C} \times \mathbf{C}$ be the unit 3-sphere and p, q two relatively prime integers. Let ζ be a primitive pth root of unity in $\mathbf{C}$. Then the cyclic group $C_p = \langle \zeta \rangle$ of order p acts on $\mathbf{S}^3$ by $\zeta.(x, y) = (\zeta x, \zeta^q y)$. Denote by P the quotient space $\mathbf{S}^3/C_p$.

Compute the homology groups, cohomology groups, and cup product structure (with integer coefficients and coefficients in $\mathbf{Z}/p\mathbf{Z}$) of P.

(2) (5 points)
The *Euler characteristic* $\chi(M)$ of a manifold (or, more generally, a compact CW complex) is the alternating sum of the dimensions of its rational homology groups:

$$\chi(M) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbf{Q}} H_i(M; \mathbf{Q}).$$

Show:

(a) The Euler characteristic of an odd-dimensional, orientable, compact manifold is zero.

(b) The Euler characteristic of an orientable, compact manifold of dimension congruent to 2 modulo 4 is even. (*Hint:* Poincaré duality and universal coefficients imply that the map

$$H^{n/2}(M; \mathbf{Q}) \otimes H^{n/2}(M; \mathbf{Q}) \xrightarrow{\cup} H^n(M; \mathbf{Q}) \cong \mathbf{Q}$$

is a nondegenerate bilinear form.)

(c) The Euler characteristic of an orientable, compact manifold of dimension divisible by 4 can be odd.