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Suggested solutions

Exam: Introduction to Finance Mathematics (MT5009), 2022-05-24

Problem 1

Relying on Capinski & Zastawniak Chapter 1 we find the following values.

(A)

$$10 \cdot e^{-0.03 \cdot \frac{1}{2}} + 10 \cdot e^{-0.03 \cdot \frac{2}{2}} + (10 + 100) \cdot e^{-0.03 \cdot \frac{3}{2}} \approx 124.7153.$$

(B) With $C = 100$, $n = 30$ and $r = 0.05$,

$$C \frac{1 - (1 + r)^{-n}}{r} \approx 1537.2451$$

Problem 2

(A) The value of the European version of the derivative is

$$\begin{aligned} H_E(0) &= \frac{1}{(1+R)} \left[p_* f(S^u) + (1 - p_*) f(S^d) \right] \\ &= \frac{1}{(1+R)} \left[\frac{R-D}{U-D} \sqrt{S^u} + \left(1 - \frac{R-D}{U-D} \right) \sqrt{S^d} \right] \\ &= 0.975. \end{aligned}$$

(B) The value of the American version of the derivative is

$$\begin{aligned} H_A(0) &= max\{f(S(0)); H_E(0)\} \\ &= max\{\sqrt{S(0)}; 0.975\} \\ &= 1. \end{aligned}$$

Problem 3

(A) The spot rates are the yields $y(0, N)$ dictated by the current prices (see Capinski & Zastawniak, pp. 247-248). We get the following equations for the yields

$$\begin{aligned} 93 &= 100e^{-y(0,1)}, \\ 103 &= 10e^{-y(0,1)} + 110e^{-2y(0,2)}, \\ 99 &= 7e^{-y(0,1)} + 7e^{-2y(0,2)} + 107e^{-3y(0,3)}. \end{aligned}$$

Hence

$$y(0,1) = -\ln(93/100) \approx 7.26\%,$$
$$y(0,2) = -\frac{1}{2}\ln\left(\frac{103 - 10e^{-y(0,1)}}{110}\right) \approx 8.02\%,$$
$$y(0,3) = -\frac{1}{3}\ln\left(\frac{99 - 7e^{-y(0,1)} - 7e^{-2y(0,2)}}{107}\right) \approx 7.08\%.$$

(B) The no-arbitrage principle implies that

$$B(0,3) = B(0,1)B(1,3),$$

where $B(t,T)$ is the price at time t of a zero-coupon unit-bond maturing at time T (see Capinski & Zastawniak, p. 249). Hence

$$B(1,3) = \frac{B(0,3)}{B(0,1)} = \frac{e^{-3y(0,3)}}{e^{-y(0,1)}} \approx 0.870 \text{ SEK}.$$

Problem 4

(A) We have

$$\mathbf{u}\mathbf{C}^{-1} = \begin{pmatrix} 20, & 40, & 100 \end{pmatrix},$$

and

$$\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top = 160,$$

hence

$$\mathbf{w}_{\text{MVP}} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top} = \left(\frac{1}{8}, \frac{2}{8}, \frac{5}{8}\right).$$

(B) The variance of the minimum variance portfolio is given by

$$\sigma^2_{\text{MVP}} = \mathbf{w}_{\text{MVP}}\mathbf{C}\mathbf{w}^\top_{\text{MVP}}.$$

The covariance matrix is

$$\mathbf{C} = \begin{pmatrix} 0.03 & 0.01 & 0 \\ 0.01 & 0.02 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}.$$

Hence

$$\sigma^2_{\text{MVP}} = 0.00625.$$

(C) We have

$$\mathbf{m} - R\mathbf{u} = \begin{pmatrix} 0.20, & 0.15, & 0.05 \end{pmatrix},$$

hence

$$(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1} = \begin{pmatrix} 5, & 5, & 5 \end{pmatrix},$$

and

$$(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}\mathbf{u}^\top = 15,$$

hence

$$\mathbf{w}_M = \frac{(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}}{(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}\mathbf{u}^\top} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

(D) Each portfolio on the minimum variance line can be obtained as a linear combination of any two portfolios on the minimum variance line with different expected returns ((see Capinski & Zastawniak, p. 77). We know that the minimum variance portfolio and the market portfolio lie on the minimum variance line, and their expected returns are

$$\mu_{\text{MVP}} = \mathbf{w}_{\text{MVP}}\mathbf{m}^\top = 0.11375,$$
$$\mu_M = \mathbf{w}_M\mathbf{m}^\top = 0.1533\ldots,$$

hence $\mu_{\text{MVP}} \neq \mu_M$ and each portfolio on the minimum variance line can be obtained from

$$\mathbf{w}_V = \alpha\mathbf{w}_{\text{MVP}} + (1 - \alpha)\mathbf{w}_M, \tag{1}$$

for some $\alpha \in \mathbb{R}$. Hence if we can find some $\alpha \in \mathbb{R}$ such that this is satisfied for the portfolio with weights $\mathbf{w}_V = (0.25, 0.3, 0.45)$, then this portfolio lies on the minimum variance line. From (1) we see that we need to find α such that

$$\alpha = \frac{w_{V,1} - w_{M,1}}{w_{\text{MVP},1} - w_{M,1}},$$
$$\alpha = \frac{w_{V,2} - w_{M,2}}{w_{\text{MVP},2} - w_{M,2}},$$
$$\alpha = \frac{w_{V,3} - w_{M,3}}{w_{\text{MVP},3} - w_{M,3}}.$$

Using $\mathbf{w}_V = (0.25, 0.3, 0.45)$ and $\mathbf{w}_{\text{MVP}}$ from (A) and $\mathbf{w}_M$ from (c), we obtain

$$\frac{w_{V,1} - w_{M,1}}{w_{\text{MVP},1} - w_{M,1}} = 0.4,$$
$$\frac{w_{V,2} - w_{M,2}}{w_{\text{MVP},2} - w_{M,2}} = 0.4,$$
$$\frac{w_{V,3} - w_{M,3}}{w_{\text{MVP},3} - w_{M,3}} = 0.4,$$

i.e. $\mathbf{w}_V = 0.4\mathbf{w}_{\text{MVP}} + 0.6\mathbf{w}_M$ and thus lies on the minimum variance line. To show that the portfolio with weights $\mathbf{w}_V$ lies on the efficient frontier we also require that $\mu_V \geq \mu_{\text{MVP}}$, which is clear since

$$\mu_V = 0.4\mu_{\text{MVP}} + 0.6\mu_M,$$

and $\mu_M > \mu_{\text{MVP}}$.

Problem 5

In order to obtain a contradiction suppose that $P_E(0) > P_A(0)$. Let us now construct an arbtrage strategy.

At time $t = 0$ (short) sell the European option and buy the American option; which generates by the contradiction assumption a positive cash flow. Do not do anything until time T at which the negative European option yields $-max\{X - S(T); 0\}$ and the American option yields $max\{X - S(T); 0\}$. Hence, the trading strategy yields exactly one net cash flow which is moreover strictly positive and we thus have an arbitrage opportunity. By the no-arbitrage principle it follows that we have a contradiction and hence the contradiction assumption does not hold and the statement in the problem has been proved.