MM7050 VT20 – Representation Theory for Finite Groups

Final Exam

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1 Instructions

- Please carefully justify your answers.
- You may e.g., use part of Problem 4 to do part of Problem 1, even if you are unsuccessful with that part of Problem 4. You may use part (a) of a problem to do part (b) even if you have not solved (a), and so on.
- Unless stated otherwise, assume that V is a finite-dimensional vector space over a field F of characteristic 0. Assume G is a finite group.

2 Grading

This exam is worth 30 points. If you completed homework assignments, your homework bonus (out of 3 points) will be added to your score. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: [26.5, 30], B: [23, 26.5), C: [19.5, 23), D: [16, 19.5), E: [12.5, 16), F: [0, 12.5).

3 General advice

- Just because the exam is written in a certain order, doesn't mean you have to solve it in that order! If you know Q4 better than Q1 and want to get it out of the way first, go for it! If you are stuck at a problem, it's usually better to move on for a bit, solve something else, and then come back to the original problem with fresh eyes.
- If you are running out of time, it is usually better to solve all problems partially (so you can hope for partial credit) instead of solving some of them perfectly and completely skipping the others. Try not to leave any problems blank!

Good luck!

4 Questions

1. Let G be the group with the presentation $G = \langle a, b : a^8 = e, a^4 = b^2, b^{-1}ab = a^{-1}\rangle$. Let $\omega = \exp(2\pi \frac{i}{8}) \in \mathbb{C}$.

   (a) (4 points) Suppose you know that $|G| = 16$ and that G has character table,

	1	**1**	**2**	**2**	**2**	**A**	**4**
	$[e]$	$[a_4]$	$[a]$	$[a_2]$	$[a_3]$	$[b]$	$[ab]$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	B	C
χ_3	1	1	-1	1	-1	1	-1
χ_4	D	1	-1	1	-1	-1	1
χ_5	2	-2	$\omega + \omega^{-1}$	0	$\omega^3 + \omega^{-3}$	0	0
χ_6	2	2	0	E	0	0	0
χ_7	2	-2	$\omega^3 + \omega^{-3}$	0	$\omega + \omega^{-1}$	0	0

Table 1: Character Table of G along with the orders of conjugacy classes

   where the numbers in bold are the orders of the conjugacy classes. Complete the character table by determining A, B, C, D, and E.

(b) (3 points) Let G' be the derived subgroup of G. Use information from the character table of G to identify which group G/G' is. Carefully justify your answer, quoting any theorems you use.

2. (3 points) Prove the transitivity of induction: if $H \subset K \subset G$ then

$$Ind_K^G Ind_H^K \psi \simeq Ind_H^G \psi$$

for any character ψ of H.

3. (3 points) Show that $G = A_4$ satisfies the hypotheses of Frobenius's Theorem (i.e. A_4 is a *Frobenius group*). Explicitly describe the set of derangements (also called the *Frobenius kernel*), K, of G. What's the group structure on K? *(i.e. can you identify K as a product of cyclic groups?)*

4. In this question, we will work over the field $F = \mathbb{R}$. Let $G = \langle x \mid x^3 = 1 \rangle$ be the cyclic group of order 3.

   (a) (2 points) Consider the homomorphism $\rho \colon G \to GL_2(\mathbb{R})$ that sends

$$\rho \colon x \mapsto -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

   where x is a generator of G. Show that it is a irreducible as a representation over $\mathbb{R}$.

   (b) (3 points) Consider the regular representation of G over $\mathbb{R}$, given by the action of G on itself by left multiplication: Let V be the 3-dimensional vector space over $\mathbb{R}$ spanned by $\{e_x, x \in G\}$, and let $g \in G$ act on e_x by e_{gx}, etc.

   Write the corresponding $\mathbb{R}[G]$-module as a direct sum of irreducible $\mathbb{R}[G]$-modules. *(Hint: What are the irreducible representations of G over $\mathbb{R}$?)*

(c) (3 points) Show that the conclusion of Schur's Lemma ('every homomorphism from an irreducible module to itself is a scalar multiple of the identity') is false if you replace $\mathbb{C}$ by $\mathbb{R}$. *(Hint: Consider a G-module homomorphism from ρ to ρ, and note that ρ is a rotation in $\mathbb{R}^2$.)*

5. Consider the subgroup S_4 of S_5 of order $4! = 24$.

   (a) (3 points) Fill in the character table for S_4. Carefully justify how you get each row, and label it properly.

	1	**6**	**8**	**6**	**3**	$\leftarrow$ sizes
	1	(12)	(123)	(1234)	(12)(34)	$\leftarrow$ conjugacy class reps

   (b) (3 points) Find all the characters of S_5 induced from the non-linear irreducible characters of S_4.

(c) (3 points) Use (b) to find the complete character table of S_5.

	1	**15**	**20**	**24**	**10**	**20**	**30**	$\leftarrow$ sizes
	1	(12)(34)	(123)	(12345)	(12)	(123)(45)	(1234)	$\leftarrow$ class reps
χ_1	1	1	1	1	1	1	1	
χ_2	1	1	1	1	-1	-1	-1	
χ_3	4	0	1	-1	2	-1	0	
χ_4	4	0	1	-1	-2	1	0	
χ_5	5							
χ_6	5							
χ_7	6							