STOCKHOLMS UNIVERSITET	Exam / Tentamen
Matematiska institutionen	MM5024 Logik 7,5 hp
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This exam consists of two parts. The basic part (grundläggande del) has 7 problems (1–7), worth a total of 20 points. The problem part (problemdel) has 3 problems (8–10), worth a total of 20 points. You can obtain a maximum of 40 points.

You may submit your answers in either English (pp. 2–3) or Swedish (pp. 4–5). In case of ambiguity the English text is the one that holds.

No aids are allowed besides paper and pen. Write clearly and motivate your answers carefully. All yes/no answers should be justified. You may use the soundness and completeness theorems (and any other theorems from the course), but state clearly when you do so.

——— Good luck! — Lycka till! ———

Written Exam (English)

Basic part

1 (3 p.) State carefully the following natural deduction rules (including any necessary conditions).

(a) $\top I$

(b) $\vee E$

(c) $\exists I$

2 (2 p.) Let $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ be the interpretations defined as follows:

i	$P_1^{\mathcal{V}_i}$	$P_2^{\mathcal{V}_i}$	$P_3^{\mathcal{V}_i}$
1	1	0	1
2	0	0	0
3	0	1	0

(a) Give a formula which holds in $\mathcal{V}_1$ and $\mathcal{V}_2$, but not in $\mathcal{V}_3$

(b) Give a formula which holds in $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$, but is not a tautology.

3 (4 p.) Derive, or give a countermodel to, each of the following:

(a) $P_1 \to P_2 \vdash P_2 \to P_1$

(b) $P_1 \to P_2 \vdash \neg P_2 \to \neg P_1$

4 (2 p.) Give the free variables of the following formulas:

(a) $(P_1(x_1, x_{47}) \to P_2(x_1)) \vee (P_2(x_1) \to \forall x_{47}\, P_1(x_{47}, x_5))$

(b) $\forall x_1\ (P_5(x_1) \to \exists x_2\, (P_5(x_2) \wedge P_{17}(x_2, x_1, x_2, x_5)))$

5 (4 p.)Are each of the following derivations correct? If not, find an error in the derivation.

(a) $\forall x_1\, P_1(x_1) \vdash_{x_0} \exists x_1\, P_1(x_1)$:

$$\dfrac{\dfrac{(x_0)\quad \forall x_1\, P_1(x_1)}{(x_0)\quad P_1(x_0)}\,\forall E}{(x_0)\quad \exists x_1\, P_1(x_1)}\,\exists I$$

(b) $\exists x_1\, P_1(x_1) \vdash \forall x_1\, P_1(x_1)$:

$$\dfrac{\exists x_1\, P_1(x_1) \qquad \dfrac{[(x_1)\quad P_1(x_1)]^1}{(x_1)\quad \forall x_1\, P_1(x_1)}\,\forall I}{\forall x_1\, P_1(x_1)}\,\exists E_1$$

6 (2 p.)Consider the theory $\Gamma := \{P_1 \vee P_2, \neg(P_1 \wedge P_2), P_1 \to P_2\}$.

(a) Is Γ consistent?

(b) Is Γ maximally consistent?

7 (3 p.)Consider the interpretation $\mathcal{A} =< \mathbb{Z}; >; + >$.

(a) Decide the arity of $\mathcal{A}$.

(b) Is the following formula true in $\mathcal{A}$? Motivate your answer

$$\varphi = \forall x_1, x_2, x_3\, (P_1(x_1, x_2) \to P_1(f_1(x_1, x_3), x_2))$$

Problem part

8 Over the arity type $\langle 2\,;\ \rangle$ (a single binary predicate), consider the closed formulas:

$$\begin{aligned}
\varphi_{\text{ref}} &\coloneqq \forall x_1\ P_1(x_1, x_1) \\
\varphi_{\text{trans}} &\coloneqq \forall x_1, x_2, x_3\ \ ((P_1(x_1, x_2) \wedge P_1(x_2, x_3)) \to P_1(x_1, x_3)) \\
\varphi_{\text{ord}} &\coloneqq \forall x_1, x_2\ \ ((P_1(x_1, x_2) \wedge P_1(x_2, x_1)) \to x_1 \doteq x_2) \\
\varphi_{\text{tot}} &\coloneqq \forall x_1, x_2\ \ (P_1(x_1, x_2) \vee P_1(x_2, x_1))
\end{aligned}$$

(a) Show that $\langle\, \mathbb{N}\,;\ \ ;\, \leq \rangle$ is a model of $\{\varphi_{\text{ref}}, \varphi_{\text{trans}}, \varphi_{\text{ord}}, \varphi_{\text{tot}}\}$.

(b) Decide whether $\varphi_{\text{ref}}, \varphi_{\text{trans}}, \vdash \varphi_{\text{ord}}$.

(c) Decide whether $\varphi_{\text{ref}}, \varphi_{\text{trans}}, \varphi_{\text{ord}} \vdash \varphi_{\text{tot}}$.

(For parts (b) and (c), justify your answer; but if you use a semantic argument (with the use of truth values) in these parts, you do not need to give full justifications of any computations of truth values you use.)

9 (a) Show that $\exists x_2\, \forall x_1\, \varphi \vdash \forall x_1\, \exists x_2\, \varphi$, for every formula φ.

(b) Give a formula φ such that $\forall x_1\, \exists x_2\, \varphi \vdash \exists x_2\, \forall x_1\, \varphi$ does not hold.

(c) Give a formula φ such that $\forall x_1\, \exists x_2\, \varphi \vdash \exists x_2\, \forall x_1\, \varphi$ holds, but $\exists x_2\, \forall x_1\, \varphi$ is not a tautology.

In each part, justify your answer.

10 The model existence lemma states: any consistent theory has some model.

The completeness theorem states: for any theory Γ and formula φ, if $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

(a) Show that the model existence lemma implies the completeness theorem.

(b) Show that the completeness theorem implies the model existence lemma.

——— End of exam ———

Skriftligt prov (Svenska)

Grundläggande del

1 (3 p.) Ange noggrant följande regler för naturlig deduktion (inklusive alla nödvändiga restriktioner).

(a) $\top I$

(b) $\vee E$

(c) $\exists I$

2 (2 p.) Låt $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ vara tolkningarna som definierats enligt följande:

i	$P_1^{\mathcal{V}_i}$	$P_2^{\mathcal{V}_i}$	$P_3^{\mathcal{V}_i}$
1	1	0	1
2	0	0	0
3	0	1	0

(a) Ge en formel som gäller i $\mathcal{V}_1$ och $\mathcal{V}_2$, men inte i $\mathcal{V}_3$

(b) Ge en formel som gäller i $\mathcal{V}_1$, $\mathcal{V}_2$, och $\mathcal{V}_3$, men som inte är en tautologi.

3 (4 p.) Härled, eller ge en motmodell till, vardera av följande:

(a) $P_1 \to P_2 \vdash P_2 \to P_1$

(b) $P_1 \to P_2 \vdash \neg P_2 \to \neg P_1$

4 (2 p.) Ange de fria variablerna i följande formler:

(a) $(P_1(x_1, x_{47}) \to P_2(x_1)) \vee (P_2(x_1) \to \forall x_{47}\, P_1(x_{47}, x_5))$

(b) $\forall x_1\ (P_5(x_1) \to \exists x_2\, (P_5(x_2) \wedge P_{17}(x_2, x_1, x_2, x_5)))$

5 (4 p.)Är de följande härdledningen korrekta? Om inte, hitta ett fel i härdledninget

(a) $\forall x_1\, P_1(x_1) \vdash_{x_0} \exists x_1\, P_1(x_1)$:

$$\dfrac{\dfrac{(x_0)\quad \forall x_1\, P_1(x_1)}{(x_0)\quad P_1(x_0)}\ \forall E}{(x_0)\quad \exists x_1\, P_1(x_1)}\ \exists I$$

(b) $\exists x_1\, P_1(x_1) \vdash \forall x_1\, P_1(x_1)$:

$$\dfrac{\exists x_1\, P_1(x_1) \qquad \dfrac{[(x_1)\quad P_1(x_1)]^1}{(x_1)\quad \forall x_1\, P_1(x_1)}\ \forall I}{\forall x_1\, P_1(x_1)}\ \exists E_1$$

6 (2 p.)Betrakta teorin $\Gamma := \{P_1 \vee P_2, \neg(P_1 \wedge P_2), P_1 \to P_2\}$.

(a) Är Γ konsistent?

(b) Är Γ maximalt konsistent?

7 (3 p.)Betrakta tolkningen $\mathcal{A} =< \mathbb{Z}; >; + >$.

(a) Bestäm aritete av $\mathcal{A}$.

(b) Är den följande formel sant i $\mathcal{A}$? Motivera ditt svar.

$$\varphi = \forall x_1, x_2, x_3 \, (P_1(x_1, x_2) \to P_1(f_1(x_1, x_3), x_2))$$

Problemdel

8 Givit aritetstypen $\langle 2 \, ; \, \rangle$ (ett enda binärt predikatsymbol), definierar vi följande slutna formler:

$$\begin{aligned}
\varphi_{\text{ref}} &\coloneqq \forall x_1 \;\; P_1(x_1, x_1) \\
\varphi_{\text{trans}} &\coloneqq \forall x_1, x_2, x_3 \;\; ((P_1(x_1, x_2) \wedge P_1(x_2, x_3)) \to P_1(x_1, x_3)) \\
\varphi_{\text{ord}} &\coloneqq \forall x_1, x_2 \;\; ((P_1(x_1, x_2) \wedge P_1(x_2, x_1)) \to x_1 \doteq x_2) \\
\varphi_{\text{tot}} &\coloneqq \forall x_1, x_2 \;\; (P_1(x_1, x_2) \vee P_1(x_2, x_1))
\end{aligned}$$

(a) Visa att $\langle \mathbb{N} \, ; \;\; ; \leq \rangle$ är en modell för $\{\varphi_{\text{ref}}, \varphi_{\text{trans}}, \varphi_{\text{ord}}, \varphi_{\text{tot}}\}$.

(b) Bestäm om $\varphi_{\text{ref}}, \varphi_{\text{trans}}, \vdash \varphi_{\text{ord}}$.

(c) Bestäm om $\varphi_{\text{ref}}, \varphi_{\text{trans}}, \varphi_{\text{ord}} \vdash \varphi_{\text{tot}}$.

(I del (b) och (c), redovisa ditt svar nogrant; men om du använder dig av semantiska metoder, behöver du inte ge fullständig redovisning för alla beräkningar av sanningvärden du använder.)

9 (a) Visa att $\exists x_2 \, \forall x_1 \, \varphi \vdash \forall x_1 \, \exists x_2 \, \varphi$, för varje formel φ.

(b) Ange en formel φ så att $\forall x_1 \, \exists x_2 \, \varphi \vdash \exists x_2 \, \forall x_1 \, \varphi$ inte gäller.

(c) Ange en formel φ så att $\forall x_1 \, \exists x_2 \, \varphi \vdash \exists x_2 \, \forall x_1 \, \varphi$ gäller, men $\exists x_2 \, \forall x_1 \, \varphi$ är inte en tautologi.

I vardera del, motivera svaret.

10 Modell-existens-lemmat ange att varje konsistent teori har en modell.

Fullständighet satsen ange att $\Gamma \vdash \varphi$ om $\Gamma \vDash \varphi$, för varje teori Γ och fomrel φ.

(a) Visa att Modell-existens-lemmat medförar

(b) Visa att medförar Modell-existens-lemmat

——— Slut på provet ———