MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Mathematics Methods for Economist MM3001
2021-11-29

Instructions:

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- Use of calculators is permitted for performing calculations. The use of graphic or programmable features is NOT permitted.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear/wrong reasoning even if the answer is correct.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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(1) (5pt) Compute the degree 3 Taylor polynomial of the function $f(x) = e^{x^2}$, around the point $x_0 = 1$, and use it to give an approximation of $f(1.1)$.
Soulution: We begin by computing the first and second derivative of $f(x)$ in $x = 1$:

$$f'(x) = 2xe^{x^2}.$$

We plug in $x = 1$ and get $f'(1) = 2e$. For the second derivative we have

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = (2 + 4x^2)e^{x^2}.$$

In $x = 1$ we get $f''(1) = 6e$. For the third derivative we have

$$f'''(x) = 8xe^{x^2} + (4x + 8x^3)e^{x^2} = (12x + 8x^3)e^{x^2}.$$

In $x = 1$ we get $f''(1) = 20e$. Now we have all the ingredients to compute the Taylor polynomial:

$$p_3(x) = e^1 + 2e(x - 1) + 3e(x - 1)^2 + \frac{20}{3}e(x - 1)^3.$$

Then we have that

$$f(1.1) \simeq p_3(1.1) = e + \frac{2e}{10} + \frac{3e}{100} + \frac{2e}{300} = \frac{371}{300}e \simeq 3.616$$

(2) The equation

$$xe^y - y^2 = 2x^3 - e^{x-1}$$

defines a curve.
(a) (1pt) Verify that point $(1, 0)$ belongs to the curve.
(b) (3pt) Compute the slope of the tangent line to the curve in $(1, 0)$.
(c) (1pt) Give the equation of the tangent line to the curve in $(1, 0)$.
Solution: For the first point we must check that the RHS and the LHS are the same when $(x, y) = (1, 0)$. We get

$$RHS = 1 - 0^2 = 1;$$

$$LHS = 2 - e^0 = 1.$$

As they are the same $(1,0)$ belongs to the curve.

The first thing to do in order to compute the slope of the curve in the given point is to derive the equation implicitly. We get

$$-2y'y + \quad e^y + xy'e^y = 6x^2 - e^{x-1}.$$

We impose again $(x,y) = (1,0)$ and get:

$$0 + \quad e^0 + y'(1)e^0 = 6 - e^0.$$

We solve for $y'(1)$ and get $y'(1) = 4$, which is exactly the slope of the curve in $(1,0)$.

Finally, for the equation of the tangent line, we now that

$$y = 4x + m,$$

since the slope is exactly the slope of the curve. We have to find m given that the tangent passes through $(1,0)$. We get $m = -4$ so the equation we are looking for is

$$y = 4x - 4.$$

Consider the function $f(x) = \frac{x^2-1}{x^2-4}$.

(a) (2pt) Give the maximum and minimum value of the function in the interval $[-1,1]$.

(b) (2pt) Find where the function is increasing or decreasing and concave or convex.

(c) (1pt) Compute $\lim_{x\to+\infty} f(x)$.

Solution: Observe that the function is continuous on its domain $D = \mathbb{R}\pm 2$, therefore it has max and min on $[-1;1]$. We look for the critical points, by computing the first derivative:

$$f'(x) = \frac{2x(x^2-4)-(x^2-1)2x}{(x^2-4)^2} = \frac{2x(x^2-4-x^2+1}{(x^2-4)^2} = \frac{-6x}{(x^2-4)^2}.$$

We have that $f'(x) = 0$ if, and only if, $x = 0$, which is an element of the interval $[-1,1]$. The candidates for extreme points are $x = \pm 1, 0$. We evaluate the functions in these points and we get:

$$f(1) = f(-1) = 0, \quad f(0) = \frac{1}{4}.$$

We deduce that the maximum value of the function in $[-1,1]$ is $\frac{1}{4}$ while the minimum value is 0.

As $(x^2-4)^2$ is always positive, the function is increasing when $x \leq 0$, $x \neq -2$ and decreasing when $x \geq 0$,$x \neq 2$. For the concavity we have to compute the second derivative:

$$\begin{aligned} f''(x) &= \frac{-6(x^2-4)^2 + 24x^2(x^2-4)^2}{(x^2-4)^4} \\ &= \frac{(x^2-4)(-6x^2-4)+24x^2)}{(x^2-4)^4} \\ &= \frac{24+18x^2}{(x^2-4)^3} \end{aligned}$$

As $24 + 18x^2$ is always positive, we have that the concavity ofthe functions depends on the sign of $(x^2-4)^3$. Thus f is concave when $x \in (-2,2)$ and convex when $x < -2$or $x > 2$.

For the last part we have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2(1-1/x^2)}{x^2(1-4/x^2)} = \lim_{x\to+\infty} \frac{1-1/x^2}{1-4/x^2} = \frac{1}{1} = 1$$

(3) Compute the following integrals:
  (a) (3pt) $\int \big(x^2\ln(x^3+1) + 5x\ln(x)\big)\,dx$,
  (b) (2pt) $\int_4^{\ln(8)} \frac{x}{6\sqrt{x^2-15}}dx$.

Solution: We begin with the first

$$\int \big(x^2\ln(x^3+1+5x\ln(x)\big)\,dx = \int x^2\ln(x^3+1)dx + \int 5x\ln(x)dx$$
$$= \frac{1}{3}\int \ln(u)du + \int 5x\ln(x)dx$$

Where in the first summand I used the substitution $u = x^3+1$, so $du = 3x^2dx$. Going forward we get

$$\int \big(x^2\ln(x^3+1+5x\ln(x)\big)\,dx = \frac{1}{3}(\ln(u)u - u) + C_1 + \frac{5}{2}x^2\ln(x) - \frac{5}{4}x^2 + C_2$$
$$= \frac{1}{3}(\ln(x^3+1)(x^3+1) - (x^3+1) + \frac{5}{2}x^2\ln(x) - \frac{5}{4}x^2 + C$$

For the second integral we set $u = x^2 - 15$. Thus $du = 2xdx$. We need to recompute the extremes of integration. In order to do that we have that if $x = 4$ then $u = 1$, and if $x = \ln(8)$, the $u = 2\ln(8) - 15$. Thus we get

$$\int_4^{\ln(8)} \frac{x}{6\sqrt{x^2-15}} = \frac{1}{12}\int_1^{\ln(8)} \frac{du}{\sqrt{u}}dx$$
$$= \frac{1}{24}[u^{\frac{1}{2}}]_1^{\ln(8)} = \frac{1}{24}(\ln(8)^{\frac{1}{2}} - 1).$$

(4) Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 \\ k-1 & k-10 & -2 \\ 3 & k+2 & 1 \end{pmatrix}$$

  (a) (2 pt) Compute the determinant of A, $|A|$ as a function of k.
  (b) (1 pt) Find all the values of k for which A is not invertible.
  (c) (2 pt) Determine whether the following linear system has 1, 0, or infinite many solutions:

$$\begin{cases} 3x & +2y & +z & = & 1 \\ -1x & -10y & -2z & = & -3 \\ 3x & 2y & +z & = & 1 \end{cases}$$

Solution: We perform elementary operation on the rows of A to make the computations of the determinant easier:

$$\begin{vmatrix} 3 & 2 & 1 \\ k-1 & k-10 & -2 \\ 3 & k+2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ k-1 & k-10 & -2 \\ 0 & k & 0 \end{vmatrix}$$
$$= (-1)(k)(-6-(k-1)) = (k)(k+5)$$

We see immediately that $\det(A) = 0$ if and only if $k = 0$ or $k = -5$, so we deduce that A is invertible if $k \neq 0, -5$.

We attempt to solve the system we perform elementary row operations to the matrix

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ -1 & 10 & -2 & -3 \\ 3 & 2 & 1 & 1 \end{array}\right)$$

We subtract the firs row to the third row:

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ -1 & 10 & -1 & -3 \\ 3 & 2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 5 & 4 & 1 & 1 \\ -7 & -4 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

As we have obtained a rows with all zero we know that the system is underdetermined. Therefore, it has infinite many solutions.

(5) Consider the two variables function

$$f(x, y) = 3x^3 - x^2 - y^2.$$

(a) (2pt) Find all the critical points of $f(x, y)$ and determine their type.
(b) (1pt) Consider now the set $D = \{(x, y) \,|x \geq 0, \text{and } x^2 + y^2 \leq 1\}$. Draw a sketch of D and determine which critical points of $f(x, y)$ lie in the interior of D.
(c) (2pt) Determine the minimum and the maximum value of $f(x, y)$ on D.

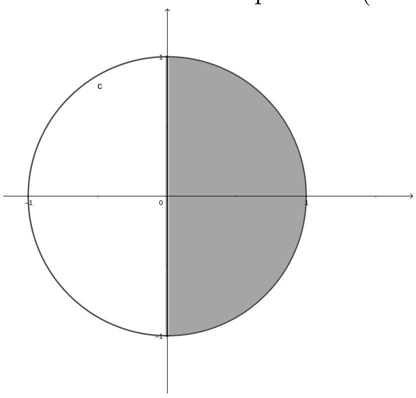
Solution: We first compute the partial derivatives of f and set them to zero to find critical points for f in the interior of D. We get

$$\frac{\partial}{\partial x} f(x, y) = 9x^2 - 2x = 0$$
$$\frac{\partial}{\partial y} f(x, y) = -2y = 0.$$

From the second equation we find $y = 0$. In the first equation we find $x(9 - 2x) = 0$ Which has as solution $x = 0$ and $x = 2/9$. This yields two critical points: $(0, 0)$ and $(2/9, 0)$. To determine the type of the critical points we have to compute the Hessian matrix

$$\mathcal{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 18x - 2 & 0 \\ 0 & -2 \end{pmatrix}$$

The determinant H of $\mathcal{H}$ is $-2(18x - 2)$ we evaluate this in the two critical points. In $(0, 0)$ H takes value 4 which is positive. As in $(0, 0)$ we have that $f_{xx} = -2$ and $f_{yy} = -2$, we deduce that $(0, 0)$ is a local maximum. In $(2/9, 0)$, H has value 2, so this is neither a minimum or a maximum point (it is a saddle point). The domain D consist in the interior of the unit circle in the first and fourth quadrant (see the gray are in the picture)



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Both critical points lies in D as $4/81 + 0 < 1$. However $(0,0)$ is not in the interior of D and should not, a priori, be considered as a candidate for max or min point. In order to evaluate the min and max value of f in D we have to investigate the restriction of f to the boundary of D. Note that the boundary has two corners $(0,-1)$ and $(0,+1)$ these have to be investigate separately. The boundary consist of two curves, the curve $x = 0$ and the curve $x^2 + y^2 = 1$. See the picture.
Curve $x = 0$. We have that $f(0,y) = -y^2$. $\frac{d}{dy} f(0,y) = -2y$ has only one critical point when $y = 0$ this yields the candidate $(0,0)$.
Curve $x^2 + y^2 = 1$. If we restrict f to this curve we obtain the function $g(x) = 3x^3 - 1$. We have that $g'(x) = 9x^2 - 1$ which yields the two critical points $x = \pm\frac{1}{3}$. As x needs to be nonnegative, we will consider only $x = 1/3$ which yields $y = \pm\frac{\sqrt{8}}{3}$. Thus we get two further candidates for the max and min points $(1/3, \sqrt{8}/3)$ and $(1/3, -\sqrt{8}/3)$.
Conclusion: From the reasoning above we get 6 candidates for max and min points of f, $(0,0)$, $(2/9,0)$, $(1/3, \pm\sqrt{8}/3)$, and $(0,\pm 1)$. We compute the values that f takes at these points (as the point $(2/9,0)$ is a saddle point we could avoind testing f in it):

$$f(0,0) = 0$$
$$f(2/9,0) = -4/243.$$
$$f(1/3, \pm\sqrt{8}/3) = 1/9 - 1 = -8/9$$
$$f(0,\pm 1) = -1$$

Of these the biggest is surely $f(0,0)$ - it is the only one which is non-negative- and the smallest is $f(0,\pm 1)$, so these are respectively the maximum and the minimum value of f in D.

FORMULAS

- Taylor polynomial of degree n of $f(x)$ around the point $x_0 = a$:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

GOOD LUCK!!!

Texten på svenska

(1) (5pt) Bestäm Taylor polynomet av grad 2 till funktionen $f(x) = e^{-\sqrt{x+3}}$ kring punkten $x_0 = 2$.

(2) Betrakta funktionen $f(x) = \frac{x^2+x+3}{x+1}$.
(a) (2pt) Antar att f defineras på $[0, 2]$: bestäm max och min värden av f samt ge var dessa värden anta.
(b) (2pt) Antar nu att f defineras på $[0, +\infty)$. Har funktionen nu en max värde? Förklara din mening.
(c) (1pt) Bestäm $\lim_{x\to+\infty} \frac{f(x)}{6x}$.

(3) Bestäm följande integraler:
(a) (3pt) $\displaystyle\int \left(x\ln(x^2+1) + 3\sqrt{x^5}\right) dx$,
(b) (2pt) $\displaystyle\int_0^1 \frac{5e^{3x}}{1-3e^{3x}}\, dx$.

(4) Betrakta serien $2 + \frac{4}{x} + \frac{8}{x^2} + \frac{16}{x^3} + \cdots$. Bestäm
(a) (2pt) for vilka x convergerar serien;
(b) (3pt) för vilka x är seriens värden 1.

(5) Låt

$$A = \begin{pmatrix} 5 & 4 & 1 \\ k-7 & k-4 & -2 \\ 5 & k-2 & 1 \end{pmatrix}$$

(a) (2 pt) Räkna determinanten till A, $|A|$ som en funktion av k.
(b) (1 pt) Hitta alla värden k sådana att A inte är invertibar.
(c) (2 pt) Använd Gausselimination för att lösa det linjär systemet nedanför

$$\begin{cases} 5x & +4y & +1z & = & 1 \\ -7x & -4y & -2z & = & -3 \\ 5x & -2y & +z & = & 1 \end{cases}$$

(6) Vi definearar området D i $\mathbb{R}^2$ som kvadraten med hörnen

$$(-1,-1), \quad (-1,1), \quad (1,-1), \quad (1,1).$$

Låt $f(x,y) = xe^{3y} - e^x$.
(a) (1pt) Skissa området D.
(b) (4pt) Bestäm max och min värde av $f(x,y)$ i D.