MATHEMATICAL INSTITUTE
STOCKHOLM UNIVERSITY
Dept. of Mathematics
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Written examination in
Mathematics III
Ordinary differential equations
August 24, 2022

Calculators are not allowed. All solutions should be motivated. Do not forget to answer all questions stated in the problem.

1 Calculate the general solution to the differential equation

$$y'' - 7y' + 12y = 6xe^x.$$

Determine the solution satisfying the initial data:

$$y(0) = 0, \;\; y'(0) = 0.$$

2 Find all solutions of the differential equation

$$x^2y'' + xy' - y = \frac{x^3}{1-x},$$

which are given by the power series at the origin $y(x) = \sum_{k=0}^{\infty} a_k x^k$.

3 Consider the following first order linear system

$$\begin{cases} \dfrac{dx}{dt} &= x + 5y, \\ \dfrac{dy}{dt} &= 2x + 4y. \end{cases}$$

a) Calculate the general solution to the system.

b) Find the solution satisfying the initial condition

$$\begin{cases} x(0) &= -5, \\ y(0) &= 2. \end{cases}$$

c) Determine whether the system is stable, asymptotically stable, or unstable.

4 Consider the autonomuous system

$$\begin{cases} \dfrac{dx}{dt} = -x + \dfrac{1}{2}\sin y \\ \dfrac{dy}{dt} = -x - 5y \end{cases}$$

(a) Determine all critical points.

(b) Determine whether these critical points are asymptotically stable?

(c) For all asymptotically stable stationary points construct a Lyapunov function.
(It is enough to consider the linearised system.)

5 Consider the differential operator

$$L = -\frac{d^2}{dx^2}$$

defined on all twice continuously differentiable functions on $[0, \pi]$ satisfying the boundary conditions

$$y(0) = 0, \quad y(\pi) = 0.$$

a) Prove that the operator L is symmetric in the space $L_2[0, \pi]$ of real valued functions with the scalar product

$$\langle u, v \rangle = \int_0^\pi u(x)v(x)dx.$$

b) Determine the spectrum and the eigenfunctions of L.

c) Let f be a C^2-function, write its expansion with respect to the eigenfunctions of L including the formula for the generalised Fourier coefficients.

GOOD LUCK!