MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avdelning Matematik
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Solution to the exam in
ODE, 7.5 hp
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1 We solve the homogeneous equation first:

$$\lambda^2 - 7\lambda + 12 = 0 \Rightarrow \lambda_{1,2} = 3, 4$$
$$\Rightarrow y_{hom} = c_1 e^{3x} + c_2 e^{4x}.$$

Particular solution to the non-homogeneous equation can be found in the form

$$y_p = (ax + b)e^x.$$

Substitution into the original equation gives:

$$2ae^x + (ax+b)e^x - 7\big(ae^x + (ax+b)e^x\big) + 12(ax+b)e^x = 6xe^t.$$

Comparing the coefficients in front of xe^x and e^x we get the linear system:

$$\begin{cases} a - 7a + 12a = 6, \\ 2a + b - 7a - 7b + 12b = 0 \end{cases} \Rightarrow a = 1, b = 1.$$

The general solution to the differential equation is

$$y = (x+1)e^x + c_1 e^{2x} + c_2 e^{3x}.$$

To satisfy the initial conditions we need to choose c_1 and c_2:

$$\begin{cases} 1 + c_1 + c_2 = 0 \\ 1 + 1 + 3c_1 + 4c_2 = 0 \end{cases} \Rightarrow c_1 = -2, c_2 = 1.$$

Summing up the solution is

$$y = (x+1)e^t - 2e^{2t} - 2e^{3t} + e^{4x}.$$

2 Assume that the solution is given by the power series: $y = \sum_{k=0}^{\infty} a_k x^k$. Then we have:

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + \dots, \\ y' &= a_1 + 2a_2 x + 3a_3 x^3 + \dots, \\ y'' &= 2a_2 + 6a_3 x + \dots \end{aligned}$$

Substitution into the differential equation gives:

$$x^2\Big(2a_2 + 6a_3 x + \dots\Big) + x\Big(a_1 + 2a_2 x + 3a_3 x^3 + \dots\Big) - \Big(a_0 + a_1 x + a_2 x^2 + \dots\Big) = x^3 + x^4 + x^5 + \dots$$

We check coefficients in front of different powers of x:

$$\begin{array}{lll} x^0: & -a_0 = 0 & \Rightarrow \quad a_0 = 0 \\ x^1: & a_1 - a_1 = 0 & \Rightarrow \quad a_1 - \text{arbitrary} \\ x^2: & 2a_2 + 2a_2 - a_2 = 0 & \Rightarrow \quad a_2 = 0 \\ x^3: & 6a_3 + 3a_3 - a_3 = 0 & \Rightarrow \quad a_3 = \frac{1}{8} \\ \vdots & & \\ x^k: & k(k-1)a_k + ka_k - a_k = 0 & \Rightarrow \quad a_k = \frac{1}{k^2-1},\ k = 3, 4, 5, \dots \end{array}$$

We get the following general solution:

$$y(x) = \alpha x + \sum_{k=3}^{\infty} \frac{1}{k^2 - 1} x^k,$$

where α is an arbitrary constant.

3 The matrix

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

has eigenvalues -1 and 6 with the eigenvectors $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. General solution to the linear system is given by

$$c_1 \begin{pmatrix} 5 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}.$$

The solution satisfying the initial conditions is

$$-\begin{pmatrix} 5 \\ -2 \end{pmatrix} e^{-t}.$$

The system is unstable since one of the eigenvalues is positive ($\lambda = 6$).

4 The equilibrium points must satisfy

$$\begin{array}{c} -x + \frac{1}{2}\sin y = 0 \\ -x - 5y = 0 \end{array}$$

implying in particular that $\sin y = -10y \Rightarrow y = 0$. Then x is calculated from any of the equations $x = 0$.

There is just one critical point $(0, 0)$.

The linearised system near this point is

$$\begin{cases} \frac{dx}{dt} &= -x + \frac{1}{2}y, \\ \frac{dy}{dt} &= -x - 5y. \end{cases}$$

The eigenvalues are

$$\lambda_{1,2} = -3 \pm \sqrt{\frac{7}{2}}$$

and both are negative, hence the system is stable near this critical point.

The Lyapunov function can be taken in the form: $L(x, y) = x^2 + by^2 > 0, (x, y) \neq (0, 0), b > 0$. Really we have

$$L_x x' + L_y y' = 2x(-x + \frac{1}{2}y) + 2y(-x - 5y) = -2x^2 + xy - 2bxy - 10by^2.$$

Choosing $b = 1/2$ we get:

$$L_x x' + L_y y' = -2x^2 - 5y^2 < 0, \quad (x, y) \neq (0, 0).$$

5 The operator is symmetric:

$$\begin{aligned} < Lu, v > &= \int_0^\pi (-u'')v dx = -u'(x)v(x)|_{x=0}^\pi + u(x)v'(x)|_{x=0}^\pi + \int_0^\pi u(-v'')dx \\ &= -u'(\pi)\underbrace{v(\pi)}_{=0} + u'(0)\underbrace{v(0)}_{=0} + \underbrace{u(\pi)}_{=0}v'(\pi) - \underbrace{u(0)}_{=0}v'(0) + < u, Lv >, \end{aligned}$$

since u, v belong to the domain and satisfy the boundary condiitions.

The eigenfunctions are solution to the equation $y'' = \lambda y, \quad \lambda = k^2$. Every solution is of the form

$$y = a \sin kx + b \cos kx.$$

Checking boundary condition at $x = 0$ we conclude that

$$y = a \sin kx.$$

The boundary condition at $x = \pi$ gives

$$\sin k\pi = 0 \Rightarrow k = n, \quad n = 1, 2, \ldots.$$

Hence the eigenvalues and the eigenfunctions are $\lambda_n = \Big(n\Big)^2$, $\psi_n = a_n \sin nx$, $n = 1, 2, 3, \ldots$ The normalisation constants a_n are calculated from

$$(a_n)^{-2} = \int_0^\pi \Big(\sin nx\Big)^2 dx = \pi/2 \Rightarrow a_n = \sqrt{\frac{2}{\pi}}.$$

We get the following eigenfunction expansion:

$$f(x) = \sum_{n=0}^{\infty} c_n \sqrt{\frac{2}{\pi}} \sin nx, \quad c_n = \int_0^\pi f(x) \sqrt{\frac{2}{\pi}} \sin nx dx,$$

which is nothing else than the standard sine Fourier transform.